

Mixing Properties for Toral Extensions of Slowly Mixing Dynamical Systems with Finite and Infinite Measure

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Abstract

We prove results on mixing and mixing rates for toral extensions of nonuniformly expanding maps with subexponential decay of correlations. Both the finite and infinite measure settings are considered. Under a Dolgopyat-type condition on nonexistence of approximate eigenfunctions, we prove that existing results for (possibly nonMarkovian) nonuniformly expanding maps hold also for their toral extensions.

1 Introduction

Dolgopyat [5] obtained results on superpolynomial decay of correlations for compact group extensions of uniformly expanding and uniformly hyperbolic dynamical systems. In this paper, we consider toral extensions of a large class of (not necessarily Markov) nonuniformly expanding maps, including the AFN maps of [25, 26], both in the finite measure setting (where the decay is subexponential) and in the infinite measure setting. Under mild hypotheses, we show that sharp mixing results for the underlying map pass over to the toral extension.

1.1 Existing results for nonuniformly hyperbolic maps

Let $f : X \rightarrow X$ be a topologically mixing map with ergodic invariant measure μ . Let $Y \subset X$ be a subset with $\mu(Y) \in (0, \infty)$. We define the first return time $\tau : Y \rightarrow \mathbb{Z}^+$ and first return map $F = f^\tau : Y \rightarrow Y$ given by

$$\tau(y) = \inf\{n \geq 1 : f^n y \in Y\}, \quad \text{and} \quad F(y) = f^{\tau(y)}(y).$$

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Under certain assumptions on F and τ , it is possible to obtain sharp mixing properties for f . More specifically, we assume that

- (i) The first return time $\tau : Y \rightarrow \mathbb{Z}^+$ is either nonintegrable with $\mu(y \in Y : \tau(y) > n) = \ell(n)n^{-\beta}$ where $\beta \in (0, 1]$ and ℓ is a slowly varying function¹, or integrable with $\mu(y \in Y : \tau(y) > n) = O(n^{-\beta})$ where $\beta > 1$.
- (ii) The first return map $F : Y \rightarrow Y$ fits into the appropriate functional abstract framework with suitable Banach space of observables $\mathcal{B}(Y) \subset L^1(Y)$ with norm $\|\cdot\|$ (see [8, 21] for the finite measure case, and [18] for the infinite measure case).

Under conditions (i) and (ii), we recall the following results for the map $f : X \rightarrow X$ and observables v_0, w_0 supported in Y with $v_0 \in \mathcal{B}(Y)$, $w_0 \in L^\infty(Y)$. Let $\bar{v}_0 = \int_Y v_0 d\mu$, $\bar{w}_0 = \int_Y w_0 d\mu$.

In the infinite measure case, define

$$\tilde{\ell}(n) = \begin{cases} \ell(n), & \beta \in (0, 1) \\ \sum_{j=1}^n \ell(j)j^{-1}, & \beta = 1 \end{cases}, \quad \text{and} \quad d_\beta = \begin{cases} \frac{1}{\pi} \sin \beta\pi, & \beta \in (0, 1) \\ 1, & \beta = 1 \end{cases}. \quad (1.1)$$

If $\beta \in (\frac{1}{2}, 1]$, then

$$\lim_{n \rightarrow \infty} \tilde{\ell}(n) \int_Y v_0 w_0 \circ f^n d\mu = d_\beta \bar{v}_0 \bar{w}_0. \quad (1.2)$$

If $\beta < \frac{1}{2}$, or if $\beta < 1$ and either $\bar{v}_0 = 0$ or $\bar{w}_0 = 0$, then

$$\int_Y v_0 w_0 \circ f^n d\mu = O(\ell(n)n^{-\beta} \|v_0\| \|w_0\|_\infty). \quad (1.3)$$

In the finite measure case, normalise so that μ is a probability measure. For all $n \geq 1$,

$$\int_Y v_0 w_0 \circ f^n d\mu - \bar{v}_0 \bar{w}_0 = \sum_{j>n} \mu(\tau > j) \bar{v}_0 \bar{w}_0 + E_\beta(n) \|v_0\| \|w_0\|_\infty, \quad (1.4)$$

where $E_\beta(n) = O(n^{-\beta})$ for $\beta > 2$, $E_\beta(n) = O(n^{-2} \log n)$ for $\beta = 2$, and $E_\beta(n) = O(n^{-(2\beta-2)})$ for $1 < \beta < 2$. Also $E_\beta(n) = O(n^{-\beta})$ for all $\beta > 1$ if $\bar{v}_0 = 0$ or $\bar{w}_0 = 0$.

Remark 1.1 The precise functional analytic hypotheses mentioned in condition (ii) play no role in this paper; we use only the consequences (1.2)–(1.4) for $\int_Y v_0 w_0 \circ f^n d\mu$. A special case is when F is a full branch Gibbs-Markov map with $\mathcal{B}(Y)$ taken to be the space $F_\theta(Y)$ of Lipschitz observables. (See Section 3 for definitions.)

¹A measurable function $\ell : (0, \infty) \rightarrow (0, \infty)$ is *slowly varying* if $\lim_{x \rightarrow \infty} \ell(\lambda x)/\ell(x) = 1$ for all $\lambda > 0$.

Prototypical examples include Pomeau-Manneville intermittent maps of the unit interval [20] such as the following:

Example 1.2 $f(x) = \begin{cases} x(1 + c_1^\gamma x^\gamma), & x \in [0, \frac{1}{2}) \\ 2x - 1, & x \in [\frac{1}{2}, 1] \end{cases}$, where $\gamma > 0$, $c_1 \in (0, 2]$. When $c_1 = 2$, the map f is Markov and was introduced in [15].

Example 1.3 $f(x) = x(1 + c_2 x^\gamma) \bmod 1$, where $\gamma > 0$, $c_2 > 0$. If c_2 is an integer, then f is Markov and belongs to the class of maps studied by [23].

In general, the above maps f are nonMarkovian and are examples of “AFN maps” [25, 26]. For all $\gamma > 0$, there is a unique (up to scaling) σ -finite invariant measure μ equivalent to Lebesgue and the measure is finite if and only if $\gamma < 1$.

In Example 1.2, it is convenient to take $Y = [\frac{1}{2}, 1]$. In Example 1.3, a convenient choice is to let Y be the domain of the right-most branch. Then $\mu(y \in Y : \tau(y) > n) \sim cn^{-\beta}$ where $\beta = 1/\gamma$ and $c > 0$, so condition (i) is satisfied. Also condition (ii) holds with $\mathcal{B}(Y)$ taken to be the space of bounded variation functions on Y .

1.2 Toral extensions

Set up In this paper, we prove analogous results for toral extensions of nonuniformly expanding maps $f : X \rightarrow X$ satisfying conditions (i) and (ii). We assume further that there exists $Z \subset Y \subset X$ (possibly $Z = Y$) with $\mu(Z) > 0$ and a return time² $\varphi : Z \rightarrow \mathbb{Z}^+$ (not necessarily a first return time) such that the return map $G = f^\varphi : Z \rightarrow Z$ satisfies

(iii) $\mu_Z(z \in Z : \varphi(z) > n) = O(n^{-(\beta-\epsilon)})$ for all $\epsilon > 0$, and $G : Z \rightarrow Z$ is a full branch Gibbs-Markov map with partition α . Moreover φ is constant on partition elements. Here μ_Z denotes the unique ergodic G -invariant probability measure absolutely continuous with respect to $\mu|_Z$.

(iv) There exists $\rho : Z \rightarrow \mathbb{Z}^+$ constant on elements of the partition α such that $G(z) = F^{\rho(z)}z$ for $z \in Z$. Moreover, $\mu_Z(z \in Z : \rho(z) > n) = O(e^{-cn})$ for some $c > 0$, and if $a \in \alpha$, then $\tau \circ F^j$ is constant on a for all $j < \rho(a)$.

(It suffices that $\mu_Z(\rho > n) = O(n^{-q})$ for q sufficiently large, depending only on β . Assumptions similar to (iv) were considered in [3].)

Given a measurable cocycle $h : X \rightarrow \mathbb{T}^d$, we form the toral extension

$$f_h : X \times \mathbb{T}^d \rightarrow X \times \mathbb{T}^d, \quad f_h(x, \psi) = (fx, \psi + h(x)).$$

The product measure $m = \mu \times d\psi$ is f_h -invariant.

² A function $\varphi : Z \rightarrow \mathbb{Z}^+$ is called a *return time* if $f^{\varphi(z)}z \in Z$ for all $z \in Z$.

It is necessary to rule out certain pathological cases, since toral extensions of mixing uniformly expanding maps need not be mixing, and mixing toral extensions can mix arbitrarily slowly. Dolgopyat [4, 5] introduced condition (v) below for proving superpolynomial decay of correlations for suspensions and compact group extensions of uniformly expanding/hyperbolic systems. Our final assumption is

(v) There do not exist approximate eigenfunctions.

The definition of approximate eigenfunctions is somewhat technical, and so is delayed until Section 4 where we show that condition (v) holds typically (in a strong sense).

Mixing results for toral extensions

Let $f : X \rightarrow X$ be a topologically mixing map with ergodic invariant measure μ and $h : X \rightarrow \mathbb{T}^d$ be a C^η cocycle, $\eta \in (0, 1]$. We consider toral extensions $f_h : X \times \mathbb{T}^d \rightarrow X \times \mathbb{T}^d$ as described previously satisfying conditions (i)–(v), where condition (ii) can be replaced by the fact that (1.2)–(1.4) hold for observables $v_0 \in \mathcal{B}(Y)$ and $w_0 \in L^\infty(Y)$.

Let $v : X \times \mathbb{T}^d \rightarrow \mathbb{R}$. For $\eta \in (0, 1)$, define $|v|_{C^\eta} = \sup_{\psi \in \mathbb{T}^d} \sup_{x \neq y} |v(x, \psi) - v(y, \psi)|/d(x, y)$ and $\|v\|_{C^\eta} = |v|_\infty + |v|_{C^\eta}$. Write $v \in C^\eta(X \times \mathbb{T}^d)$ if $\|v\|_{C^\eta} < \infty$.

For $\eta \in (0, 1)$ and $p \in \mathbb{N}$, write $v \in C^{\eta,p}(X \times \mathbb{T}^d)$ if v is p -times differentiable with respect to ψ with derivatives that lie in $C^\eta(X \times \mathbb{T}^d)$, and set $\|v\|_{C^{\eta,p}} = \sum_{|j| \leq p} \|\frac{\partial^j v}{\partial \psi^j}\|_{C^\eta}$.³

For our main results, we consider observables v, w supported in $Y \times \mathbb{T}^d$. Let $v_0(y) = \int_{\mathbb{T}^d} v(y, \psi) d\psi$. Suppose that $v_0 \in \mathcal{B}(Y)$, $v - v_0 \in C^{\eta,p}(Y \times \mathbb{T}^d)$, $w \in L^\infty(Y \times \mathbb{T}^d)$, where $p \in \mathbb{N}$ is chosen sufficiently large, and write $\|v\| = \|v_0\| + \|v - v_0\|_{C^{\eta,p}}$. Let $\bar{v} = \int_{Y \times \mathbb{T}^d} v dm$, $\bar{w} = \int_{Y \times \mathbb{T}^d} w dm$.

Theorem 1.4 *In the infinite measure case, define $\tilde{\ell}$ and d_β as in (1.1).*

(a) *Suppose that $\beta \in (\frac{1}{2}, 1]$. Then*

$$\lim_{n \rightarrow \infty} \tilde{\ell}(n) n^{1-\beta} \int_{Y \times \mathbb{T}^d} v w \circ f_h^n dm = d_\beta \bar{v} \bar{w}.$$

(b) *Suppose either that $\beta \in (0, \frac{1}{2}]$, or that $\beta \in (0, 1]$ and either $\bar{v} = 0$ or $\bar{w} = 0$. Then for all $\epsilon > 0$,*

$$\int_{Y \times \mathbb{T}^d} v w \circ f_h^n dm = O(n^{-(\beta-\epsilon)} \|v\| \|w\|_\infty).$$

³ Given $j \in \mathbb{Z}^d$ with $j_1, \dots, j_d \geq 0$, we write $|j| = j_1 + \dots + j_d$ and $\frac{\partial^j}{\partial \psi^j} = \frac{\partial^{|j|}}{\partial \psi_1^{j_1} \dots \partial \psi_d^{j_d}}$.

Remark 1.5 Under stronger conditions on $\mu(\tau > n)$, improved error rates and higher order asymptotics are obtained for nonuniformly expanding maps f in [18, 22]. These results apply in particular to the maps considered in [15] and extend to the toral case.

Theorem 1.6 *In the finite measure case, for all $\epsilon > 0$,*

$$\int_{Y \times \mathbb{T}^d} v w \circ f_h^n dm - \bar{v} \bar{w} = \sum_{j > n} \mu(\tau > j) \bar{v} \bar{w} + O(n^{-q} \|v\| \|w\|_\infty),$$

where $q = \beta - \epsilon$ if $\beta \geq 2$ and $q = 2\beta - 2$ if $1 < \beta < 2$. We can also take $q = \beta - \epsilon$ if $\beta > 1$ and $\bar{v} = 0$ or $\bar{w} = 0$.

Strategy of the proofs For L^2 observables $v, w : X \times \mathbb{T}^d \rightarrow \mathbb{R}$, we write

$$v(x, \psi) = \sum_{k \in \mathbb{Z}^d} v_k(x) e^{ik \cdot \psi}, \quad (1.5)$$

where $v_k \in L^2(X, \mathbb{C})$, $v_{-k} = \bar{v}_k$, and similarly for w . Conditions (i) and (ii) above on the first return map $F = f^\tau : Y \rightarrow Y$ take care of the zero Fourier modes v_0 and w_0 , so the main contribution of the current paper is to deal with the nonzero modes. In Section 2, we show how this can be achieved under conditions (iii)–(v) using the induced map $G = f^\varphi : Z \rightarrow Z$.

Remark 1.7 If the first return map $F = f^\tau : Y \rightarrow Y$ is a full branch Gibbs-Markov map, then there is no need for a second inducing scheme: we can simply take $G = F$. (Conditions (iii) and (iv) can now be ignored.) Even here our results are new. This simplified set up applies to the maps in Examples 1.2 and 1.3 if they are Markov, and more generally to the class of Thaler maps [23].

For the nonMarkovian “AFN” maps of [25, 26], we use both of the inducing schemes and our main theorems apply with $\mathcal{B}(Y)$ taken to be the space of bounded variation functions on Y . This includes all cases in Examples 1.2 and 1.3.

Upper bounds on decay of correlations In the finite measure case, we also obtain an upper bound for decay of correlations, see Corollary 2.6. This is simpler than the other results mentioned here, and we need only to use one inducing scheme, $G = f^\varphi : Z \rightarrow Z$, satisfying condition (iii) with $\beta > 1$. In particular, our result applies to toral extensions of maps modelled by Young towers with polynomial tails and summable decay of correlations [24], and shows under condition (v) that the toral extension f_h mixes at the same rate as f .

The remainder of the paper is structured as follows. In Section 2, we state results, Theorems 2.2 and 2.3, on the nonzero Fourier modes in (1.5) and use these to prove the results from the introduction. In Section 3, we recall the definition and basic properties of the Gibbs-Markov induced map $G = f^\varphi$. In Section 4, we recall the notions of eigenfunctions and approximate eigenfunctions. In Section 5, we recall

some standard results about smoothness of Fourier series. In Section 6, we obtain some estimates for twisted transfer operators corresponding to the induced dynamics on Y , and we derive a Dolgopyat-type estimate. In Section 7, we obtain estimates for certain associated renewal operators. Theorems 2.2 and 2.3 are proved in Sections 8 and 9 respectively.

Notation We use “big O” and \ll notation interchangeably, writing $a_n = O(b_n)$ or $a_n \ll b_n$ if there is a constant $C > 0$ such that $a_n \leq Cb_n$ for all $n \geq 1$.

2 Reduction to the nonzero Fourier modes

In this section, we show how to reduce to dealing with the nonzero Fourier modes in (1.5). First, we require the following basic expansion of $\int_{X \times \mathbb{T}^d} v w \circ f_h^n dm$. Note that $f_h^n(x, \psi) = (f^n x, \psi + h_n(x))$ where $h_n = \sum_{j=0}^{n-1} h \circ f^j$.

Proposition 2.1 *Let $v, w : X \times \mathbb{T}^d \rightarrow \mathbb{R}$ be L^2 observables with Fourier series as in (1.5). Then $\int_{X \times \mathbb{T}^d} v w \circ f_h^n dm = \sum_{k \in \mathbb{Z}^d} \int_X e^{ik \cdot h_n} v_{-k} w_k \circ f^n d\mu$ for all $n \geq 0$.*

Proof Expanding into Fourier series,

$$\begin{aligned} \int_{X \times \mathbb{T}^d} v w \circ f_h^n dm &= \sum_{j,k \in \mathbb{Z}^d} \int_{X \times \mathbb{T}^d} v_j(x) e^{ij \cdot \psi} w_k(f^n x) e^{ik \cdot (\psi + h_n(x))} dm \\ &= \sum_{j,k \in \mathbb{Z}^d} \int_X v_j(x) w_k(f^n x) e^{ik \cdot h_n(x)} d\mu \int_{\mathbb{T}^d} e^{i(j+k) \cdot \psi} d\psi = \sum_{k \in \mathbb{Z}^d} \int_X v_{-k}(x) w_k(f^n x) e^{ik \cdot h_n(x)}, \end{aligned}$$

as required. ■

The next two results concern the nonzero Fourier modes

$$S_{v,w}(n) = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \int_X e^{ik \cdot h_n} v_{-k} w_k \circ f^n d\mu.$$

Theorem 2.2 *Assume that the induced map $G = f^\varphi : Z \rightarrow Z$ and the C^η cocycle $h : X \rightarrow \mathbb{T}^d$ satisfy conditions (iii)–(v).*

Then there exists $p \in \mathbb{N}$ such that for all observables v, w supported in $Y \times \mathbb{T}^d$ with $v \in C^{\eta,p}(Y \times \mathbb{T}^d)$, $w \in L^\infty(Y \times \mathbb{T}^d)$, and for all $\epsilon > 0$,

$$S_{v,w}(n) = O(n^{-(\beta-\epsilon)} \|v - v_0\|_{C^{\eta,p}} |w|_\infty).$$

Theorem 2.3 *Let $h : X \rightarrow \mathbb{T}^d$ be a C^η cocycle, $\eta \in (0, 1]$, and assume nonexistence of approximate eigenfunctions. Let $\varphi : Z \rightarrow \mathbb{Z}^+$ be a (general) return time such that $\mu_Z(\varphi > n) = O(n^{-\beta})$ where $\beta > 1$, and $G = f^\varphi : Z \rightarrow Z$ is full branch Gibbs-Markov.*

Then there exists $p \in \mathbb{N}$ such that $S_{v,w}(n) = O(n^{-(\beta-1)} \|v - v_0\|_{C^{\eta,p}} |w|_\infty)$ for all observables v, w with $v \in C^{\eta,p}(X \times \mathbb{T}^d)$, $w \in L^\infty(X \times \mathbb{T}^d)$.

Remark 2.4 We say that $v : X \times \mathbb{T}^d \rightarrow \mathbb{R}$ is a *trigonometric polynomial* if only finitely many of the Fourier coefficients $v_k : X \rightarrow \mathbb{C}$ in (1.5) are nonzero.

If at least one of the observables v, w is a trigonometric polynomial, then all of our results simplify. Instead of requiring nonexistence of approximate eigenfunctions, we require only the nonexistence of ordinary eigenfunctions (see Section 4.1). Moreover, we can take $p = 0$.

Remark 2.5 In the case of trigonometric polynomials, Theorem 2.3 recovers and improves upon [2] where similar results are obtained for $\beta > 2$. The improved convergence rate for observables supported in Y in Theorem 2.2 was not obtained in [2].

All of our results about toral extensions f_h are immediate consequences of Theorems 2.2 and 2.3 combined with known results for f . In particular, in the proofs of Theorem 1.4 and Theorem 1.6 below we use (1.2)–(1.4), while in the upper bounds result on decay of correlation, namely Corollary 2.6 below, we use the result of Young [24].

Proof of Theorem 1.4 Write $\int_{Y \times \mathbb{T}^d} v w \circ f_h^n dm = \int_Y v_0 w_0 \circ f^n d\mu + S_{v,w}(n)$. For $\beta > \frac{1}{2}$, by (1.2), $\lim_{n \rightarrow \infty} \tilde{\ell}(n) n^{1-\beta} \int_Y v_0 w_0 \circ f^n d\mu = d_\beta \bar{v}_0 \bar{w}_0 = d_\beta \bar{v} \bar{w}$. By Theorem 2.2, $\tilde{\ell}(n) n^{1-\beta} S_{v,w}(n) = O(n^{1-2\beta+2\epsilon} \|v - v_0\|_{C^{\eta,p}} |w|_\infty)$. Since $\beta > \frac{1}{2}$ and ϵ is arbitrarily small, part (a) follows.

For $\beta \in (0, \frac{1}{2}]$, or if $\bar{v}_0 = 0$ or $\bar{w}_0 = 0$, by (1.3), $\int_Y v_0 w_0 \circ f^n d\mu = O(n^{-(\beta-\epsilon)} \|v_0\| \|w_0\|_\infty)$. Hence part (b) follows from Theorem 2.2. ■

Proof of Theorem 1.6 Write $\int_{Y \times \mathbb{T}^d} v w \circ f_h^n dm - \bar{v} \bar{w} = g(n) + S_{v,w}(n)$, where $g(n) = \int_Y v_0 w_0 \circ f^n d\mu - \bar{v}_0 \bar{w}_0$. By (1.4),

$$g(n) = \sum_{j>n} \mu(\tau > j) \bar{v}_0 \bar{w}_0 + E_\beta(n) \|v_0\| \|w_0\|_\infty = \sum_{j>n} \mu(\tau > j) \bar{v} \bar{w} + E_\beta(n) \|v_0\| \|w_0\|_\infty.$$

The result follows from the estimates for $E_\beta(n)$ together with the estimates in Theorem 2.2 for $S_{v,w}(n)$. ■

Corollary 2.6 Let $h : X \rightarrow \mathbb{T}^d$ be a C^η cocycle, $\eta \in (0, 1]$, and assume nonexistence of approximate eigenfunctions. Let $\varphi : Z \rightarrow \mathbb{Z}^+$ be a (general) return time such that $\mu_Z(\varphi > n) = O(n^{-\beta})$ where $\beta > 1$, and $G = f^\varphi : Z \rightarrow Z$ is full branch Gibbs-Markov.

Then there exists $p \in \mathbb{N}$ such that

$$|\int_{X \times \mathbb{T}^d} v w \circ f_h^n dm - \int_{X \times \mathbb{T}^d} v dm \int_{X \times \mathbb{T}^d} w dm| = O(n^{-(\beta-1)} \|v\|_{C^{\eta,p}} |w|_\infty),$$

for all $v \in C^{\eta,p}(X \times \mathbb{T}^d)$, $w \in L^\infty(X \times \mathbb{T}^d)$.

Proof Write

$$\int_{X \times \mathbb{T}^d} v w \circ f_h^n dm - \int_{X \times \mathbb{T}^d} v dm \int_{X \times \mathbb{T}^d} w dm = \int_X v_0 w_0 \circ f^n d\mu - \int_X v_0 d\mu \int_X w_0 d\mu + S_{v,w}(n).$$

By Young [24],

$$|\int_X v_0 w_0 \circ f^n d\mu - \int_X v_0 d\mu \int_X w_0 d\mu| \leq C n^{-(\beta-1)} \|v_0\|_{C^\eta} |w_0|_\infty,$$

for all v_0 Hölder and w_0 in L^∞ . Hence the result follows from Theorem 2.3. \blacksquare

3 Induced Gibbs-Markov maps

Let (X, d) be a locally compact separable bounded metric space with Borel measure μ_0 and let $f : X \rightarrow X$ be a nonsingular transformation for which μ_0 is ergodic. Let $Z \subset X$ be a measurable subset with $\mu_0(Z) \in (0, \infty)$, and let α be an at most countable measurable partition of Z . We assume that there is a return time function $\varphi : Z \rightarrow \mathbb{Z}^+$, constant on each $a \in \alpha$ with value $\varphi(a) \geq 1$, and constants $\lambda > 1$, $\eta \in (0, 1]$, $C_1 \geq 1$, such that for each $a \in \alpha$,

- (1) $G = f^{\varphi(a)} : a \rightarrow Z$ is a measure-theoretic bijection.
- (2) $d(Gz, Gz') \geq \lambda d(z, z')$ for all $z, z' \in a$.
- (3) $d(f^\ell z, f^\ell z') \leq C_1 d(Gz, Gz')$ for all $z, z' \in a$, $0 \leq \ell < \varphi(a)$.
- (4) $g_a = \log \frac{d(\mu_0|_{a \circ G^{-1}})}{d\mu_0|_Z}$ satisfies $|g_a(z) - g_a(z')| \leq C_1 d(z, z')^\eta$ for all $z, z' \in Z$.

The induced map $G = f^\varphi : Z \rightarrow Z$ is uniformly expanding and there is a unique G -invariant probability measure μ_Z on Z equivalent to $\mu_0|_Z$ with density bounded above and below. Moreover μ_Z is mixing. This leads to a unique (up to scaling) f -invariant measure μ on X equivalent to μ_0 , see for example [24, Theorem 1]. An explicit definition of μ is given in Remark 7.1. Properties (1) and (4) are the defining conditions for the induced map $G : Z \rightarrow Z$ to be a *full branch Gibbs-Markov map*.

We assume throughout that $\gcd\{\varphi(a) : a \in \alpha\} = 1$. Then f is topologically mixing, and in the finite measure case μ is mixing.

If $a_0, \dots, a_{n-1} \in \alpha$, we define the n -cylinder $[a_0, \dots, a_{n-1}] = \bigcap_{j=0}^{n-1} G^{-j} a_j$. Let $\theta \in (0, 1)$ and define the symbolic metric $d_\theta(z, z') = \theta^{s(z, z')}$ where the *separation time* $s(z, z')$ is the greatest integer $n \geq 0$ such that z and z' lie in the same n -cylinder. In the remainder of this section, we fix $\theta \in [\lambda^{-\eta}, 1)$. For convenience we rescale the metric d on X so that $\text{diam}(Z) \leq 1$.

Proposition 3.1 $d(z, z')^\eta \leq d_\theta(z, z')$ for all $z, z' \in Z$.

Proof Let $n = s(z, z')$. By condition (2),

$$1 \geq \text{diam } Z \geq d(G^n z, G^n z') \geq \lambda^n d(z, z') \geq (\theta^{1/\eta})^{-n} d(z, z').$$

Hence $d(z, z')^\eta \leq \theta^n = d_\theta(z, z')$. \blacksquare

An observable $v : Z \rightarrow \mathbb{R}$ is *Lipschitz* if $\|v\|_\theta = |v|_\infty + |v|_\theta < \infty$ where $|v|_\theta = \sup_{z \neq z'} |v(z) - v(z')|/d_\theta(z, z')$. The set $F_\theta(Z)$ of Lipschitz observables is a Banach space. More generally, we say that $v : Z \rightarrow \mathbb{R}$ is *locally Lipschitz*, and write $v \in F_\theta^{\text{loc}}(Z)$, if $v|_a \in F_\theta(a)$ for each $a \in \alpha$. Accordingly, we define $D_\theta v(a) = \sup_{z, z' \in a: z \neq z'} |v(z) - v(z')|/d_\theta(z, z')$.

We say that an observable $v : Z \rightarrow \mathbb{R}^d$ lies in $F_\theta(Z, \mathbb{R}^d)$ if $v_1, \dots, v_d \in F_\theta(Z)$, and we define $|v|_\theta = \max_{j=1, \dots, d} |v_j|_\theta$ and $\|v\|_\theta = \max_{j=1, \dots, d} \|v_j\|_\theta$. Similarly, we define $F_\theta^{\text{loc}}(Z, \mathbb{R}^d)$ and $D_\theta v(a)$.

Proposition 3.2 *Let $h : X \rightarrow \mathbb{T}^d$ be a C^η cocycle. Define the induced cocycle $H(z) = \sum_{\ell=0}^{\varphi(z)-1} h(f^\ell z)$. Then $H \in F_\theta^{\text{loc}}(Z, \mathbb{T}^d)$ for all $\omega \in [0, 2\pi]$, and there is a constant $C_2 \geq 1$ such that*

$$D_\theta H(a) \leq C_2 |h|_{C^\eta} \varphi(a),$$

for all $\omega \in [0, 2\pi]$, $a \in \alpha$.

Proof Let $z, z' \in a$. Then $\varphi(z) = \varphi(z') = \varphi(a)$. Let $C'_1 = C_1^\eta$. By condition (3) and Proposition 3.1,

$$\begin{aligned} |H(z) - H(z')| &\leq \sum_{\ell=0}^{\varphi(a)-1} |h(f^\ell z) - h(f^\ell z')| \leq |h|_{C^\eta} \sum_{\ell=0}^{\varphi(a)-1} d(f^\ell z, f^\ell z')^\eta \\ &\leq C'_1 |h|_{C^\eta} \varphi(a) d(Gz, Gz')^\eta \leq C'_1 |h|_{C^\eta} \varphi(a) d_\theta(Gz, Gz') = C'_1 \theta^{-1} |h|_{C^\eta} \varphi(a) d_\theta(z, z'), \end{aligned}$$

yielding the required estimate for $D_\theta H(a)$. \blacksquare

The transfer operator $R : L^1(Z) \rightarrow L^1(Z)$ corresponding to the induced map $G : Z \rightarrow Z$ is given by $\int_Z Rv w d\mu_Z = \int_Z v w \circ G d\mu_Z$ for all $v \in L^1(Z)$, $w \in L^\infty(Z)$. It can be easily seen that $(Rv)(z) = \sum_{a \in \alpha} e^{g(z_a)} v(z_a)$ where z_a denotes the unique preimage of z in a under G and g is the potential. Similarly, $(R^n v)(z) = \sum_{a \in \alpha_n} e^{g_n(z_a)} v(z_a)$ where z_a denotes the unique preimage of z in a under G^n and $g_n(z) = \sum_{j=0}^{n-1} g(G^j z)$. Moreover, there exists a constant C_3 such that

$$e^{g_n(z)} \leq C_3 \mu_Z(a), \quad \text{and} \quad |e^{g_n(z)} - e^{g_n(z')}| \leq C_3 \mu_Z(a) d_\theta(G^n z, G^n z'), \quad (3.1)$$

for all $z, z' \in a$, $a \in \alpha_n$, $n \geq 1$.

Proposition 3.3 *There exists $\tau \in (0, 1)$ such that $\|R^n v - \int_Z v d\mu_Z\|_\theta \leq C \tau^n \|v\|_\theta$, for all $n \geq 1$ and $v \in F_\theta(Z)$.*

Proof This follows from the quasicompactness [1, Section 4.7] of the transfer operator R . \blacksquare

4 Eigenfunctions and approximate eigenfunctions

In this section, we recall the notion of approximate eigenfunction, and show that typically there are none. That is, condition (v) in the introduction holds typically.

In Subsection 4.1, we consider ordinary eigenfunctions as mentioned in Remark 2.4. (Non-existence of eigenfunctions is a sufficient condition for a technical result on renewal operators, namely Proposition 7.2, required in the proof of our main results.) Approximate eigenfunctions are then considered in Subsection 4.2.

Throughout this section, we work with toral extensions of a map $f : X \rightarrow X$ with full branch Gibbs-Markov induced map $G = f^\varphi : Z \rightarrow Z$ corresponding to a general return time $\varphi : Z \rightarrow \mathbb{Z}^+$. Given a measurable cocycle $h : X \rightarrow \mathbb{T}^d$, we define the induced cocycle $H : Z \rightarrow \mathbb{T}^d$ given by $H(z) = \sum_{\ell=0}^{\varphi(z)-1} h(f^\ell z)$.

4.1 Eigenfunctions

In this subsection, we define eigenfunctions and recall some of their basic properties. Let S^1 denote the unit circle in \mathbb{C} .

Definition 4.1 A measurable function $v : Z \rightarrow S^1$ is an *eigenfunction* if there exist frequencies $k \in \mathbb{Z}^d \setminus \{0\}$ and $\omega \in [0, 2\pi)$ such that $v \circ G = e^{ik \cdot H} e^{i\omega \varphi} v$.

By Remark 2.4, nonexistence of eigenfunctions is a sufficient condition for our main results in the case of trigonometric polynomials. The next result shows that nonexistence of eigenfunctions is typical.

Proposition 4.2 Suppose that h is C^η for some $\eta > 0$ and that z_1 and z_2 are fixed points for $G : Z \rightarrow Z$. If there exists an eigenfunction, then there exist $k_1, k_2 \in \mathbb{Z}^d \setminus \{0\}$ such that $k_1 \cdot H(z_1) = k_2 \cdot H(z_2) \bmod 2\pi$.

Proof Suppose that v is an eigenfunction with frequencies $k \in \mathbb{Z}^d \setminus \{0\}$ and $\omega \in [0, 2\pi)$. Since G is Gibbs-Markov, it follows by Livšic regularity that v is continuous. Since z_j are fixed points, we obtain $e^{ik \cdot H(z_j)} e^{i\omega \varphi(z_j)} = 1$. Hence $e^{i\varphi(z_2)k \cdot H(z_1)} e^{i\omega \varphi(z_1)\varphi(z_2)} = 1$ and $e^{i\varphi(z_1)k \cdot H(z_2)} e^{i\omega \varphi(z_1)\varphi(z_2)} = 1$. It follows that $e^{i\varphi(z_2)k \cdot H(z_1)} = e^{i\varphi(z_1)k \cdot H(z_2)}$. The result follows with $k_1 = \varphi(z_2)k$ and $k_2 = \varphi(z_1)k$. ■

It follows that nonexistence of eigenfunctions holds generically (for a residual set of C^η cocycles $h : X \rightarrow \mathbb{T}^d$ for any fixed $\eta > 0$). An open and dense criterion is given in Proposition 4.6 below.

4.2 Approximate eigenfunctions

For $k \in \mathbb{Z}^d$, $\omega \in [0, 2\pi]$, define $M_{k,\omega} : L^\infty(Z) \rightarrow L^\infty(Z)$,

$$M_{k,\omega} v = e^{-ik \cdot H} e^{-i\omega \varphi} v \circ G.$$

Note that v is an eigenfunction with frequencies k , ω if and only if $M_{k,\omega} v = v$.

Definition 4.3 There are *approximate eigenfunctions on a subset* $Z_0 \subset Z$ if there exist constants $\xi, \zeta > 0$ arbitrarily large and $C \geq 1$, and sequences

$$u_j \in F_\theta(Z), \quad k_j \in \mathbb{Z}^d \setminus \{0\}, \quad \omega_j \in [0, 2\pi), \quad \chi_j \in [0, 2\pi), \quad n_j = [\zeta \ln |k_j|] \in \mathbb{N}, \quad j \geq 1,$$

with $\lim_{j \rightarrow \infty} |k_j| = \infty$, $|u_j| \equiv 1$ and $|u_j|_\theta \leq C|k_j|$, such that

$$|(M_{k_j, \omega_j}^{n_j} u_j)(z) - e^{i\chi_j} u_j(z)| \leq C|k_j|^{-\xi},$$

for all $z \in Z_0$ and all $j \geq 1$.

Definition 4.4 A subset $Z_\infty \subset Z$ is called a *finite subsystem* of Z if $Z_\infty = \bigcap_{n \geq 1} G^{-n} Z_0$ where Z_0 is the union of finitely many elements from the partition α .

Definition 4.5 We say that *there exist approximate eigenfunctions* if for every finite subsystem $Z_\infty \subset Z$ there exist approximate eigenfunctions on Z_∞ .

Proposition 4.6 Suppose that $f : X \rightarrow X$ is (piecewise) C^r , $r \geq 2$. Let Z_∞ be a finite subsystem corresponding to a union Z_0 of at least two partition elements. Then there are no approximate eigenfunctions on Z_∞ for a C^2 open and C^r dense set of C^r cocycles $h : X \rightarrow \mathbb{T}^d$ for all $r \geq 2$.

The proof of this result is postponed until Appendix A. It follows that the results in this paper are valid for open and dense sets of smooth toral extensions.

5 Fourier analysis and Hölder norms

In this section, we recall some standard results about smoothness of Fourier series [14]. Let A_n be a sequence of bounded linear operators on some Banach space \mathcal{X} and set $A(\omega) = \sum_{n=1}^\infty A_n e^{in\omega}$, $\omega \in [0, 2\pi]$. If $A \in L^1$ then we define the Fourier coefficients $\hat{A}_n = (1/2\pi) \int_0^{2\pi} e^{-in\omega} A(\omega) d\omega$.

When speaking of regularity of A , we regard A as a 2π -periodic function on \mathbb{R} . Let $|A|_{C^0} = \sup_\omega \|A(\omega)\|$. For $m \in \mathbb{N}$, define $\|A\|_{C^m} = \max_{j=0, \dots, m} |A^{(j)}|_{C^0}$. For $q = m + \alpha$, $m \in \mathbb{N}$, $\alpha \in [0, 1)$, define $\|A\|_{C^q} = \|A\|_{C^m} + |A^{(m)}|_\alpha$ where $|A|_\alpha = \sup_{\omega_1 \neq \omega_2} |A(\omega_1) - A(\omega_2)| / |\omega_1 - \omega_2|^\alpha$.

Proposition 5.1 Suppose that $\sum_{j > n} \|A_j\| \leq Cn^{-q}$ for constants $C \geq 1$, $q > 0$, where q is not an integer. Then there is a universal constant D_q depending only on q such that $A : [0, 2\pi] \rightarrow L(\mathcal{X}, \mathcal{X})$ is C^q and $\|A\|_{C^q} \leq CD_q$.

Proof The details are written out for example in [2, Lemma 2.4]. ■

Proposition 5.2 Suppose that $A : [0, 2\pi] \rightarrow L(\mathcal{X}, \mathcal{X})$ is C^q , $q > 0$. Then there is a universal constant D_q depending only on q such that $\|\hat{A}_n\| \leq D_q \|A\|_{C^q} n^{-q}$.

Proof The details are written out for example in [2, Lemma 2.5]. ■

Remark 5.3 If $q > 1$ in Propositions 5.1 or 5.2, then $A_n = \hat{A}_n$ and the Fourier series is uniformly absolutely convergent.

Next, we consider Hölder norms of families of operator functions $A, B : [0, 2\pi] \rightarrow L(\mathcal{X}, \mathcal{X})$ where $A(\omega)$ is invertible for all $\omega \in [0, 2\pi]$ and $B(\omega) = A(\omega)^{-1}$.

Lemma 5.4 *For each $m \in \mathbb{N}$, there is a universal constant $c_m > 0$ such that for all $q = m + \alpha$, $\alpha \in [0, 1)$,*

$$\|B\|_{C^q} \leq c_m(1 + \|B\|_{C^0})^{2q+2}(1 + \|A\|_{C^q})^{2q+1}.$$

Proof First we consider the case $q = m \in \mathbb{N}$. The case $m = 0$ is trivial. For $m \geq 1$, note that $D^m B$ is a linear combination of terms of the form $(D^{n_1} B)(D^{n_2} A)(D^{n_3} B)$ with $n_1 + n_2 + n_3 = m$ and $n_2 \geq 1$. Inductively,

$$\begin{aligned} |D^m B|_{C^0} &\leq c'_m \sum_{\substack{n_1+n_2+n_3=m \\ n_2 \geq 1}} |D^{n_1} B|_{C^0} |D^{n_2} A|_{C^0} |D^{n_3} B|_{C^0} \leq c'_m \|A\|_{C^m} \sum_{n_1+n_3 \leq m-1} \|B\|_{C^{n_1}} \|B\|_{C^{n_3}} \\ &\leq c''_m \|A\|_{C^m} \sum_{n_1+n_3 \leq m-1} (1 + \|B\|_{C^0})^{2n_1+2n_3+4} (1 + \|A\|_{C^m})^{2n_1+2n_3+2} \\ &\leq c'''_m (1 + \|B\|_{C^0})^{2m+2} (1 + \|A\|_{C^m})^{2m+1}, \end{aligned}$$

establishing the required result when $q = m$ is an integer.

When $q = m + \alpha$, we have in addition that

$$\begin{aligned} |D^m B|_\alpha &\leq c'_m \sum_{\substack{n_1+n_2+n_3=m \\ n_2 \geq 1}} (2|D^{n_1} B|_\alpha |D^{n_2} A|_{C^0} |D^{n_3} B|_{C^0} + |D^{n_1} B|_{C^0} |D^{n_2} A|_\alpha |D^{n_3} B|_{C^0}) \\ &\leq 3c'_m \|A\|_{C^q} \sum_{n_1+n_3 \leq m-1} \|B\|_{C^{n_1+\alpha}} \|B\|_{C^{n_3}} \\ &\leq c''_m \|A\|_{C^q} \sum_{n_1+n_3 \leq m-1} (1 + \|B\|_{C^0})^{2n_1+2\alpha+2n_3+4} (1 + \|A\|_{C^q})^{2n_1+2\alpha+2n_3+2} \\ &\leq c'''_m (1 + \|B\|_{C^0})^{2q+2} (1 + \|A\|_{C^q})^{2q+1}, \end{aligned}$$

completing the proof. ■

6 Estimates for induced twisted transfer operators

Throughout this section, we assume condition (iii) on the induced map $G = f^\varphi : Z \rightarrow Z$. Recall from Section 3 that R is the transfer operator corresponding to G , and that $H : Z \rightarrow \mathbb{T}^d$ is the induced cocycle $H(z) = \sum_{\ell=0}^{\varphi(z)-1} h(f^\ell y)$.

For $k \in \mathbb{Z}^d$, define the twisted transfer operators $R_k : L^1(Z) \rightarrow L^1(Z)$, $R_k v = R(e^{ik \cdot H} v)$. We can write $R_k = \sum_{n=1}^{\infty} R_{k,n}$ where $R_{k,n} : L^1(Z) \rightarrow L^1(Z)$ is given by

$$R_{k,n} v = R_k(1_{\{\varphi=n\}} v).$$

Define $R_k(\omega) : L^1(Z) \rightarrow L^1(Z)$ for $\omega \in [0, 2\pi]$ by setting

$$R_k(\omega) v = \sum_{n=1}^{\infty} R_{k,n} e^{in\omega} v = R(e^{ik \cdot H} e^{i\omega \varphi} v).$$

Note that $R_k(\omega)^n v = R^n(e^{ik \cdot H_n} e^{i\omega \varphi_n} v)$ where $H_n = \sum_{j=0}^{n-1} H \circ G^j$, $\varphi_n = \sum_{j=0}^{n-1} \varphi \circ G^j$.

In Subsection 6.1, we derive some basic estimates for the operators $R_{k,n}$ and $R_k(\omega)$. In Subsection 6.2, we obtain a Dolgopyat-type estimate.

6.1 Some basic estimates

Proposition 6.1 *For all $k \in \mathbb{Z}^d$, $\omega \in [0, 2\pi]$, $n \geq 1$, (a) $|R_{k,n}|_{\infty} \leq C_3 \mu_Z(\varphi = n)$ and (b) $|R_k(\omega)|_{\infty} \leq 1$.*

Proof Let $z \in Z$. For each $a \in \alpha$, let z_a be the unique preimage $z_a \in a \cap G^{-1}(z)$. Then

$$(R_{k,n} v)(z) = \sum_{a \in \alpha: \varphi(a)=n} e^{g(z_a)} e^{ik \cdot H(z_a)} v(z_a).$$

By (3.1),

$$|R_{k,n} v|_{\infty} \leq C_3 \sum_{a \in \alpha: \varphi(a)=n} \mu_Z(a) |v|_{\infty} = C_3 \mu_Z(\varphi = n) |v|_{\infty},$$

proving part (a).

Since $|R|_{\infty} = 1$ and $R_k(\omega) v = R(e^{ik \cdot H} e^{i\omega \varphi} v)$, part (b) is immediate. \blacksquare

Lemma 6.2 *Let $\epsilon > 0$ and fix $\theta \in [\lambda^{-\eta\epsilon}, 1)$. There exists a constant $C \geq 1$ such that for every $v \in F_{\theta}(Z)$, $k \in \mathbb{Z}^d \setminus \{0\}$, $\omega \in [0, 2\pi]$, and for every n -cylinder $a \in \alpha_n$, $n \geq 1$,*

$$|R_k(\omega)^n(1_a v)|_{\theta} \leq C \mu_Z(a) \left\{ |k|^{\epsilon} \sum_{j=0}^{n-1} \theta^{n-j} \varphi(G^j a)^{\epsilon} |v|_{\infty} + \theta^n |v|_{\theta} \right\}.$$

Proof Let $z \in Z$, and let z_a be the unique preimage $z_a \in a \cap G^{-n}(z)$. Noting that φ_n is constant on a ,

$$(R_k(\omega)^n(1_a v))(z) = e^{g_n(z_a)} e^{ik \cdot H_n(z_a)} e^{i\omega \varphi_n(a)} v(z_a),$$

and

$$(R_k(\omega)^n(1_a v))(z) - (R_k(\omega)^n(1_a v))(z') = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= (e^{g_n}(z_a) - e^{g_n}(z'_a))e^{ik \cdot H_n(z_a)}e^{i\omega\varphi_n(a)}v(z_a), \\ I_2 &= e^{g_n}(z'_a)(e^{ik \cdot H_n(z_a)} - e^{ik \cdot H_n(z'_a)})e^{i\omega\varphi_n(a)}v(z_a), \\ I_3 &= e^{g_n}(z'_a)e^{ik \cdot H_n(z'_a)}e^{i\omega\varphi_n(a)}(v(z_a) - v(z'_a)). \end{aligned}$$

By (3.1),

$$|I_1| \leq C_3\mu_Z(a)|v|_\infty d_\theta(z, z'), \quad |I_3| \leq C_3\mu_Z(a)|v|_\theta d_\theta(z_a, z'_a) = C_3\theta^n\mu_Z(a)|v|_\theta d_\theta(z, z').$$

Using the inequality $|e^{ix} - 1| \leq 2|x|^\epsilon$ for all $x \in \mathbb{R}$, $\epsilon \in (0, 1]$,

$$|I_2| \leq 2C_3\mu_Z(a)|k|^\epsilon |H_n(z_a) - H_n(z'_a)|^\epsilon |v|_\infty.$$

Let $\gamma \in [\lambda^{-\eta}, 1)$. By definition of z_a, z'_a ,

$$d_\gamma(G^j z_a, G^j z'_a) = \gamma^{-j} d_\gamma(z_a, z'_a) = \gamma^{n-j} d_\gamma(z, z'),$$

for $j = 0, \dots, n-1$, and so by Proposition 3.2 (with $\theta = \gamma$),

$$|H(G^j z_a) - H(G^j z'_a)| \leq D_\gamma H(G^j a) d_\gamma(G^j z_a, G^j z'_a) \leq C_2 |h|_{C^\eta} \varphi(G^j a) \gamma^{n-j} d_\gamma(z, z').$$

Hence

$$|H_n(z_a) - H_n(z'_a)| = \left| \sum_{j=0}^{n-1} (H(G^j z_a) - H(G^j z'_a)) \right| \leq C_2 |h|_{C^\eta} \sum_{j=0}^{n-1} \gamma^{n-j} \varphi(G^j a) \gamma^{s(z, z')}.$$

It follows that

$$\begin{aligned} |I_2| &\leq 2C_2 C_3 |k|^\epsilon |v|_\infty \mu_Z(a) |h|_{C^\eta}^\epsilon \left| \sum_{j=0}^{n-1} \gamma^{n-j} \varphi(G^j a) \right|^\epsilon \gamma^{\epsilon s(z, z')} \\ &\leq 2C_2 C_3 |k|^\epsilon |v|_\infty \mu_Z(a) |h|_{C^\eta}^\epsilon \sum_{j=0}^{n-1} \gamma^{\epsilon(n-j)} \varphi(G^j a)^\epsilon \gamma^{\epsilon s(z, z')}. \end{aligned}$$

Choosing $\gamma = \theta^{1/\epsilon}$,

$$|I_2| \leq 2C_2 C_3 |k|^\epsilon |v|_\infty \mu_Z(a) |h|_{C^\eta}^\epsilon \sum_{j=0}^{n-1} \theta^{n-j} \varphi(G^j a)^\epsilon d_\theta(z, z').$$

Combining the estimates for I_1, I_2, I_3 yields the required result. ■

Corollary 6.3 *Choose ϵ such that $\varphi^\epsilon \in L^1(Z)$ and let $\theta \in [\lambda^{-\eta\epsilon}, 1)$. There exists a constant $C_4 \geq 1$ (depending on h, φ, ϵ) such that for every $\theta \in (0, 1)$, $v \in F_\theta(Z)$, $k \in \mathbb{Z}^d \setminus \{0\}$, $\omega \in [0, 2\pi]$, $n \geq 1$,*

$$(a) \quad \|R_{k,n}\|_\theta \leq C_4 \mu_Z(\varphi = n) |k|^\epsilon n^\epsilon.$$

$$(b) \quad |R_k(\omega)^n v|_\theta \leq C_4 \{|k|^\epsilon |v|_\infty + \theta^n |v|_\theta\}.$$

Proof Taking $\omega = 0$, $n = 1$, $a \in \alpha$ in Lemma 6.2, we obtain

$$|R_k(1_a v)|_\theta \leq C \mu_Z(a) \{|k|^\epsilon \varphi(a)^\epsilon |v|_\infty + |v|_\theta\} \leq C \mu_Z(a) |k|^\epsilon \varphi(a)^\epsilon \|v\|_\theta.$$

Summing over those a with $\varphi(a) = n$, we obtain that $|R_{k,n} v|_\theta \ll \mu_Z(\varphi = n) |k|^\epsilon n^\epsilon \|v\|_\theta$. This combined with Proposition 6.1(a) yields part (a).

To prove part (b), we write $R_k(\omega)^n v = \sum_{a \in \alpha_n} R_k(\omega)^n (1_a v)$ and sum the estimates from Lemma 6.2. Note that

$$\begin{aligned} \sum_{a \in \alpha_n} \mu_Z(a) \sum_{j=0}^{n-1} \theta^{n-j} \varphi(G^j a)^\epsilon &= \sum_{j=0}^{n-1} \theta^{n-j} \sum_{b \in \alpha_{n-j}} \varphi(b)^\epsilon \sum_{a \in \alpha_n: G^j a = b} \mu_Z(a) \\ &= \sum_{j=0}^{n-1} \theta^{n-j} \sum_{b \in \alpha_{n-j}} \varphi(b)^\epsilon \mu_Z(b) \leq \theta(1 - \theta)^{-1} \sum_{b \in \alpha} \mu_Z(b) \varphi(b)^\epsilon. \end{aligned}$$

$$\text{Hence } |R_k(\omega)^n v|_\theta \leq C \{\theta(1 - \theta)^{-1} |k|^\epsilon \sum_{a \in \alpha} \mu_Z(a) \varphi(a)^\epsilon |v|_\infty + \theta^n |v|_\theta\}. \quad \blacksquare$$

Corollary 6.4 Choose $\epsilon \in (0, \beta)$ so that $\beta - \epsilon$ is not an integer and such that $\varphi^\epsilon \in L^1(Z)$. Let $\theta \in [\lambda^{-\eta_\epsilon}, 1)$.

For each $k \in \mathbb{Z}^d \setminus \{0\}$, the map $R_k : [0, 2\pi] \rightarrow L(F_\theta(Z), F_\theta(Z))$, $\omega \mapsto R_k(\omega)$, is $C^{\beta-\epsilon}$. Moreover, there is a constant $C \geq 1$ independent of k such that $\|R_k\|_{C^{\beta-\epsilon}} \leq C |k|^\epsilon$.

Proof Recall from condition (iii) that $\mu_Z(\varphi > n) = O(n^{-(\beta-\epsilon)})$. By Corollary 6.3(a), we have that $\sum_{j>n} \|R_{k,j}\|_\theta \ll |k|^\epsilon n^{-(\beta-2\epsilon)}$. Now apply Proposition 5.1. \blacksquare

6.2 A Dolgopyat-type estimate

The argument in this subsection is a direct adaptation of an argument in [16] and is included mainly for completeness. Propositions 6.6 and 6.7 below correspond to [16, Lemmas 3.12 and 3.13] respectively, and the Dolgopyat-type estimate, Lemma 6.8, follows immediately.

Throughout, we fix $\epsilon \in (0, 1]$ such that $\varphi^\epsilon \in L^1(Z)$, and $\theta \in [\lambda^{-\eta_\epsilon}, 1)$.

Remark 6.5 As in [4, Section 6], we define $\|v\|_k = \max\{|v|_\infty, |v|_\theta / (2C_4 |k|^\epsilon)\}$. Then it follows from Proposition 6.1(a) and Corollary 6.3(a) that $\|R_k(\omega)^n\|_k \leq C_4 + \frac{1}{2}$ for all $n \geq 1$. Moreover, $\|R_k(\omega)^n\|_k \leq 1$ for all $n \geq n_0$ (where $n_0 = \lfloor \ln(2C_4) / (-\ln \theta) \rfloor + 1$).

Since we are estimating operator norms with respect to $\|\cdot\|_k$, we consider the unit ball $F_\theta(Z)_k = \{v \in F_\theta : \|v\|_k \leq 1\}$. It follows from Remark 6.5 that $|R_k(\omega)^n v|_\infty \leq 1$ and $|R_k(\omega)^n v|_\theta \leq 2C_4|k|^\epsilon$ for all $v \in F_\theta(Z)_k$ and $n \geq n_0$.

Throughout, Z_0 denotes a fixed subset of Z consisting of a finite union of partition elements of Z , and $Z_\infty = \bigcap_{j \geq 0} G^{-j} Z_0$. Note that the potential g is uniformly bounded on Z_∞ and moreover $g_n(z) \leq n|1_{Z_\infty} g|_\infty$ for all $z \in Z_\infty$ and $n \geq 1$.

Proposition 6.6 *Let $\xi_2, \zeta_0 > 0$. Then there exist $\xi_1 > 0$ and $\zeta > \zeta_0$, such that the following is true for each fixed $|k| \geq 2$, $\omega \in [0, 2\pi]$, setting $n(k) = \lceil \zeta \ln |k| \rceil$:*

Suppose that there exists $v_0 \in F_\theta(Z)_k$ such that for all $x \in Z_\infty$ and all $j = 0, 1, 2$,

$$|(R_k(\omega)^{j n(k)} v_0)(x)| \geq 1 - 1/|k|^{\xi_1}.$$

Then there exists $w \in F_\theta(Z)$ with $|w| \equiv 1$, $|w|_\theta \leq 16C_4|k|$, and $\varphi \in [0, 2\pi)$ such that for all $z \in Z_\infty$,

$$|(M_{b,\omega}^{n(k)} w)(z) - e^{i\chi} w(z)| \leq 8/|k|^{\xi_2}.$$

Proof We write $n = n(k)$ and $\tilde{C}_4 = 16C_4$. Set

$$\zeta = (\xi_2 + 2 + \ln \tilde{C}_4 / \ln 2) / (-\ln \theta), \quad \xi_1 = \max\{1, 2\xi_2 + \zeta|1_{Z_\infty} p|_\infty\}.$$

If necessary, increase ζ so that $\zeta > \zeta_0$. Following [4, Section 8], we write $v_j = R_k(\omega)^{j n} v_0$ and $v_j = s_j w_j$, where $|w_j(x)| \equiv 1$ and $1 - 1/|k|^{\xi_1} \leq s_j(x) \leq 1$ for $x \in Z_\infty$. Note that $|v_j|_\theta \leq 2C_4|k|^\epsilon$ so that $|w_j|_\theta \leq \tilde{C}_4|k|^\epsilon \leq \tilde{C}_4|k|$. Rearrange $v_1 = R_k(\omega)^n v_0$ to obtain $w_1^{-1} R_k(\omega)^n (s_0 w_0) = s_1 \geq 1 - 1/|k|^{\xi_1}$. It then follows from the definition of $R_k(\omega)$ that $e^{g_n(z)} [1 - \Re(e^{ik \cdot H_n(z)} w_0(z) w_1^{-1}(G^n z))] \leq 1/|k|^{\xi_1}$ for all $z \in Z$ with $G^n z \in Z_\infty$. Hence $|e^{ik \cdot H_n(z)} w_0(z) - w_1(G^n z)| \leq 2(e^{-g_n(z)}/|k|^{\xi_1})^{1/2}$. Similarly, with w_0 and w_1 replaced by w_1 and w_2 . Restricting to $z \in Z_\infty$, we have $e^{-g_n(z)}/|k|^{\xi_1} \leq 1/|k|^{2\xi_2}$ and hence

$$|e^{ik \cdot H_n(z)} w_0(z) - w_1(G^n z)| \leq 2/|k|^{\xi_2}, \quad |e^{ik \cdot H_n(z)} w_1(z) - w_2(G^n z)| \leq 2/|k|^{\xi_2}, \quad (6.1)$$

for all $z \in Z_\infty$. Fix $q \in Z_\infty$ and define $w_j(q) = e^{i\chi_j}$ for $j = 0, 1$. Let $\chi = \chi_0 - \chi_1$. To each z , we associate $z^* = q_0 \cdots q_{n-1} z_n z_{n+1} \cdots \in Z_\infty$. Then z^* is within distance θ^n of q and $G^n z^* = G^n z$. We obtain

$$\begin{aligned} |e^{ik \cdot H_n(z^*)} e^{i\chi_0} - w_1(G^n z)| &\leq 2/|k|^{\xi_2} + \tilde{C}_4|k|\theta^n \leq 3/|k|^{\xi_2} \\ |e^{ik \cdot H_n(z^*)} e^{i\chi_1} - w_2(G^n z)| &\leq 2/|k|^{\xi_2} + \tilde{C}_4|k|\theta^n \leq 3/|k|^{\xi_2} \end{aligned}$$

(by the choice of ζ), and so $|e^{-i\chi} w_1(G^n z) - w_2(G^n z)| \leq 6/|k|^{\xi_2}$. Substituting into (6.1) yields the required approximate eigenfunction $w = w_1$. \blacksquare

Proposition 6.7 *For any $\xi_1, \zeta > 0$, there exists $\xi > 0$ and $C \geq 1$ with the following property.*

Let $|k| \geq 1$ and suppose that for any $v \in F_\theta(Z)_k$ there exists $x_0 \in Z_\infty$ and $j \leq \lceil \zeta \ln |k| \rceil$ such that $|(R_k(\omega)^j v)(x_0)| \leq 1 - 1/|k|^{\xi_1}$. Then $\|(I - R_k(\omega))^{-1}\|_k \leq C|k|^\xi$.

Proof Following [4, Section 7], we use the pointwise estimate on iterates of $R_k(\omega)$ to obtain estimates on the L^1 , L^∞ and $\|\cdot\|_k$ norms.

Write $\hat{u} = R_k(\omega)^j v$ and $u = R_k(\omega)^{\ell(k)} v$ where $\ell(k) = [\zeta \ln |k|]$. Note that $|\hat{u}|_\infty \leq 1$ and $|\hat{u}|_\theta \leq 2C_4|k|^\epsilon \leq 2C_4|k|$. Hence, $|\hat{u}(x)| \leq 1 - 1/(2|k|^{\xi_1})$ for all x within distance $1/(4C_4|k|^{\xi_1+1})$ of x_0 . Call this subset U . If \mathcal{C}_m is an m -cylinder, then $\text{diam } \mathcal{C}_m = \theta^m$, so provided $\theta^m < 1/(4C_4|k|^{\xi_1+1})$, the m -cylinder containing x_0 lies inside U . It suffices to take $m \sim (\xi_1 + 1) \ln |k| / (-\ln \theta)$. By (3.1),

$$\mu_Z(U) \geq \mu_Z(\mathcal{C}_m) \geq C_3^{-1} e^{-g_m(x_0)} \geq C_3^{-1} e^{-m|1_{Z_\infty} g|_\infty} \geq C^{-1} |k|^{-(\xi_1+1)\xi_2},$$

where $\xi_2 = |1_{Z_\infty} g|_\infty / (-\ln \theta)$. Breaking up Z into U and $Z \setminus U$,

$$|u|_1 \leq |\hat{u}|_1 \leq (1 - 1/(2|k|^{\xi_1}))\mu_Z(U) + 1 - \mu_Z(U) = 1 - \mu_Z(U)/(2|k|^{\xi_1}) \leq 1 - C^{-1}|k|^{-\xi_3},$$

where $\xi_3 = \xi_1 + \xi_2 + \xi_1 \xi_2$. By Proposition 3.3,

$$\begin{aligned} |R_k(\omega)^n u|_\infty &\leq |(R^n |u|)|_\infty \leq |(R^n |u| - \int |u|)|_\infty + |u|_1 \ll \tau^n \|u\|_\theta + |u|_1 \\ &\leq (1 + 2C_4|k|)\tau^n + 1 - C^{-1}|k|^{-\xi_3}. \end{aligned}$$

Choosing $n = n_1(k) = [\zeta_1 \ln |k|]$ where $\zeta_1 \gg 1$ ensures that

$$|R_k(\omega)^{\ell(k)+n_1(k)} v|_\infty = |R_k(\omega)^{n_1(k)} u|_\infty \leq 1 - C^{-1}|k|^{-\xi_3}.$$

Setting $n_2(k) = [\zeta_2 \ln |k|]$ where $\zeta_2 = \zeta + \zeta_1$,

$$|R_k(\omega)^{n_2(k)} v|_\infty \leq 1 - C^{-1}|k|^{-\xi_3}.$$

By Proposition 6.1(a) and Corollary 6.3(b), $|R_k(\omega)^{n_2(k)+n}|_\infty \leq 1 - C^{-1}|k|^{-\xi_3}$ for all $n \geq 0$, and

$$|R_k(\omega)^{n_2(k)+n} v|_\theta / (2C_4|k|^\epsilon) \leq \frac{1}{2} + \theta^n C_4 \leq \frac{3}{4},$$

for n sufficiently large (independent of k). Increasing ζ_2 slightly, we obtain $\|R_k(\omega)^{n_2(k)} v\|_k \leq 1 - C^{-1}|k|^{-\xi_3}$. Hence $\|(I - R_k(\omega)^{n_2(k)})^{-1}\|_k \leq C|k|^{\xi_3}$. Using the identity $(I - A)^{-1} = (I + A + \cdots + A^{m-1})(I - A^m)^{-1}$ and Remark 6.5, we obtain

$$\|(I - R_k(\omega))^{-1}\|_k = O(n_2(k)|k|^{\xi_3}) = O(|k|^\xi),$$

for any choice of $\xi > \xi_3$. ■

Lemma 6.8 *Assume conditions (iii) and (v). Then there exists $\xi > 0$ and $C \geq 1$ such that $\|(I - R_k(\omega))^{-1}\|_\theta \leq C|k|^\xi$ for all $k \in \mathbb{Z}^d \setminus \{0\}$ and all $\omega \in [0, 2\pi]$.*

Proof This is immediate from Propositions 6.6 and 6.7. ■

7 Renewal operators

Throughout, we fix $\epsilon \in (0, \beta]$ such that $\beta - \epsilon$ is not an integer, $\varphi^\epsilon \in L^1(Z)$, and $\theta \in [\lambda^{-\eta_\epsilon}, 1)$.

Define the tower $\Delta = \{(z, \ell) \in Z \times \mathbb{Z} : 0 \leq \ell \leq \varphi(z) - 1\}$. The tower map $\hat{f} : \Delta \rightarrow \Delta$ is given by $\hat{f}(z, \ell) = \begin{cases} (z, \ell + 1), & \ell \leq \varphi(z) - 2 \\ (Gz, 0), & \ell = \varphi(z) - 1 \end{cases}$, with ergodic \hat{f} -invariant measure $\mu_\Delta = \mu_Z \times \text{counting}$. Let $L : L^1(\Delta) \rightarrow L^1(\Delta)$ denote the transfer operator corresponding to $\hat{f} : \Delta \rightarrow \Delta$. (So $\int_\Delta L v d\mu_\Delta = \int_\Delta v w \circ \hat{f} d\mu_\Delta$.)

Denote by $\pi : \Delta \rightarrow X$ the projection $\pi(z, \ell) = f^\ell z$.

Remark 7.1 Since π is a semiconjugacy from \hat{f} to f , the measure $\mu = \pi_* \mu_\Delta$ is an ergodic f -invariant measure on X . This is the measure described in Section 3.

Given a cocycle $h : X \rightarrow \mathbb{T}^d$, we define the lifted cocycle $\hat{h} = h \circ \pi : \Delta \rightarrow \mathbb{T}^d$. For $k \in \mathbb{Z}^d$, define the twisted transfer operators $L_k : L^1(\Delta) \rightarrow L^1(\Delta)$, $L_k v = L(e^{ik \cdot \hat{h}} v)$.

Next, define the renewal operators $T_{k,n} : L^1(Z) \rightarrow L^1(Z)$ given by $T_{k,0} = I$ and for $n \geq 1$,

$$T_{k,n} v = 1_Z L_k^n(1_Z v) = 1_Z L^n(1_Z e^{ik \cdot \hat{h}_n} v).$$

Define $T_k(\omega) : L^1(Z) \rightarrow L^1(Z)$ for $\omega \in [0, 2\pi]$,

$$T_k(\omega) = \sum_{n=0}^{\infty} T_{k,n} e^{in\omega}.$$

Note that $G = \hat{f}^\varphi : Z \rightarrow Z$ is the first return to Z for the map $\hat{f} : \Delta \rightarrow \Delta$. Hence for all $k \in \mathbb{Z}^d$ we have the renewal equation,

$$T_k(\omega) = (I - R_k(\omega))^{-1}.$$

Let $\hat{T}_{k,n}$ denote Fourier coefficients of $T_k(\omega)$.

Since the expression $S_{v,w}(n)$ in Theorem 2.2 is a sum over $k \in \mathbb{Z}^d \setminus \{0\}$, we restrict attention throughout to this range of k . (The operators $T_{0,n}$ and $T_0(\omega)$ were studied in [8, 21, 18].)

Proposition 7.2 *Assume condition (iii) and nonexistence of eigenfunctions. Then $T_{k,n} = \hat{T}_{k,n}$ for all $k \in \mathbb{Z}^d \setminus \{0\}$, $n \geq 0$.*

For $\beta > 1$, this follows from Remark 5.3 using the estimate $\hat{T}_{k,n} = O(n^{-(\beta-\epsilon)})$, and the assumption that there are no eigenfunctions is not required. (The case $\beta > 2$ was treated similarly in [2].) The proof of Proposition 7.2 for general $\beta > 0$ is postponed to Appendix B.

Lemma 7.3 *Assume conditions (iii) and (v). Choose ϵ and θ as in Corollary 6.4. Then there are constants $C \geq 1$, $\xi > 0$, such that*

$$\|T_{k,n}\| \leq C|k|^\xi n^{-(\beta-\epsilon)},$$

for all $k \in \mathbb{Z}^d \setminus \{0\}$, $n \geq 1$.

Proof By Corollary 6.4, $\omega \mapsto R_k(\omega)$ is $C^{\beta-\epsilon}$. By Lemma 6.8, $I - R_k(\omega)$ is invertible and so $\omega \mapsto T_k(\omega) = (I - R_k(\omega))^{-1}$ is $C^{\beta-\epsilon}$. Hence by Propositions 5.2 and 7.2,

$$\|T_{k,n}\| = \|\hat{T}_{k,n}\| \ll \|T_k\|_{C^{\beta-\epsilon}} n^{-(\beta-\epsilon)}.$$

By Lemma 5.4 and Corollary 6.4,

$$\|T_k\|_{C^{\beta-\epsilon}} \ll \|R_k\|_{C^{\beta-\epsilon}}^{2\beta+1} \|(I - R_k)^{-1}\|_{C^0}^{2\beta+2} \ll |k|^{\epsilon(2\beta+1)} \sup_{\omega \in [0, 2\pi]} \|(I - R_k(\omega))^{-1}\|^{2\beta+2}.$$

Hence by Lemma 6.8, $\|T_k\|_{C^{\beta-\epsilon}} \ll |k|^\xi$. The result follows. \blacksquare

8 Proof of Theorem 2.2

In this section, we assume conditions (iii)–(v). Let $f : X \rightarrow X$ with induced map Gibbs-Markov map $G = f^\varphi : Z \rightarrow Z$ as in Section 3. Let μ_Z denote the associated ergodic G -invariant measure on Z .

As before, we fix $\epsilon \in (0, \beta]$ such that $\beta - \epsilon$ is not an integer, $\varphi^\epsilon \in L^1(Z)$, and $\theta \in [\lambda^{-\eta_\epsilon}, 1)$.

Let $\hat{f} : \Delta \rightarrow \Delta$ be the tower map defined in Section 4.1 with ergodic \hat{f} -invariant measure $\mu_\Delta = \mu_Z \times \text{counting}$. We continue to let $\pi : \Delta \rightarrow X$ denote the semiconjugacy $\pi(z, \ell) = f^\ell z$ from \hat{f} to f . Recall that $\pi_* \mu_\Delta = \mu$ is the underlying ergodic f -invariant measure on X . Given a cocycle $h : X \rightarrow \mathbb{T}^d$, we define the lifted cocycle $\hat{h} = h \circ \pi : \Delta \rightarrow \mathbb{T}^d$.

The symbolic metric d_θ on Z defined in Section 3 extends to a metric on Δ by defining $d_\theta((z, \ell), (z', \ell')) = \begin{cases} d_\theta(z, z'), & \ell = \ell' \\ 1 & \ell \neq \ell' \end{cases}$. An observable $v : \Delta \rightarrow \mathbb{R}$ is Lipschitz if $\|v\|_\theta = |v|_\infty + |v|_\theta < \infty$ where $|v|_\theta = \sup_{p \neq q} |v(p) - v(q)|/d_\theta(p, q) < \infty$. Let $F_\theta(\Delta)$ denote the space of Lipschitz observables on Δ .

Proposition 8.1 *If $v \in C^\eta(X)$, then $\hat{v} = v \circ \pi \in F_\theta(\Delta)$. Moreover, there is a constant $C \geq 1$ such that $\|\hat{v}\|_\theta \leq C\|v\|_{C^\eta}$.*

Proof Clearly, $|\hat{v}|_\infty \leq |v|_\infty$. Let $q = (z, \ell)$, $q' = (z', \ell') \in \Delta$. If $\ell \neq \ell'$, we have $|\hat{v}(q) - \hat{v}(q')| \leq 2|v|_\infty = 2|v|_\infty d_\theta(q, q')$. If $\ell = \ell'$, then setting $C'_1 = C_1^\eta$, and using

condition (3) in the definition of nonuniformly expanding map and Proposition 3.1,

$$\begin{aligned} |\hat{v}(q) - \hat{v}(q')| &= |v(f^\ell z) - v(f^\ell z')| \leq |v|_{C^\eta} d(f^\ell y, f^\ell z')^\eta \leq |v|_{C^\eta} C'_1 d(Gz, Gz')^\eta \\ &\leq |v|_{C^\eta} C'_1 d_\theta(Gz, Gz') = |v|_{C^\eta} C'_1 \theta^{-1} d_\theta(z, z'). \end{aligned}$$

Hence $|\hat{v}|_\theta \ll \|v\|_{C^\eta}$. ■

In Theorem 2.2, we are interested in observables $v : X \rightarrow \mathbb{R}$ supported in Y . These lift to observables $\hat{v} : \Delta \rightarrow \mathbb{R}$ supported in $\hat{Y} = \pi^{-1}(Y)$. Proposition 8.1 guarantees that if $v \in C^\eta(Y)$, then $\hat{v} \in F_\theta(\hat{Y})$.

Proposition 8.2 *Let $a \in \alpha$, $0 \leq \ell < \varphi(a)$. If $(a \times \{\ell\}) \cap \hat{Y} \neq \emptyset$, then $a \times \{\ell\} \subset \hat{Y}$.*

Proof Suppose there exists $z_0 \in a$ such that $(z_0, \ell) \in \hat{Y}$. Then there exists $q \geq 1$ such that $\tau_q(z_0) = \ell$. Note that $\tau_q = \ell < \varphi = \tau_\rho$, so $q < \rho$ and τ_q is constant on a by condition (iv). Hence $\tau_q(z) = \ell$ for all $z \in a$, and it follows that $a \times \{\ell\} \subset \hat{Y}$. ■

The tower Δ can be partitioned into levels $\{\Delta_n; n \geq 0\}$ and diagonals $\{D_n; n \geq 1\}$ where

$$\Delta_n = \{(z, n) \in Z \times \mathbb{N} : \varphi(z) > n\}, \quad D_n = \{(z, \varphi(z) - n) \in Z \times \mathbb{Z} : \varphi(z) > n\}.$$

Note that $\mu_\Delta(\Delta_n) = \mu_\Delta(D_n) = \mu_Z(\varphi > n)$. We have the corresponding partitions $\hat{Y} \cap \Delta_n$ and $\hat{Y} \cap D_n$ of \hat{Y} .

Proposition 8.3 $\sum_{j \geq n} \mu_\Delta(\hat{Y} \cap \Delta_j) = O(n^{-(\beta-\epsilon)}), \sum_{j \geq n} \mu_\Delta(\hat{Y} \cap D_j) = O(n^{-(\beta-\epsilon)}).$

Proof The proof of these estimates is based on [3].

First notice that both $\bigcup_{j \geq n} \hat{Y} \cap \Delta_j$ and $\bigcup_{j \geq n} \hat{Y} \cap D_j$ are contained in $\{(z, \ell) \in \hat{Y} : \varphi(z) > n\}$, so it suffices to show that $\mu_\Delta(\{(z, \ell) \in \hat{Y} : \varphi(z) > n\}) = O(n^{-(\beta-\epsilon)})$.

Next, we write $\{(z, \ell) \in \hat{Y} : \varphi(z) > n\} = \bigcup_{q=1}^\infty \{(z, \ell) \in \hat{Y} : \varphi(z) > n, \rho(z) = q\}$. If $\rho(z) = q$, then $\varphi(z) = \tau_q(z)$ and so there are precisely q values of $\ell \in \{0, 1, \dots, \varphi(z) - 1\}$ such that $(z, \ell) \in \hat{Y}$. Hence

$$\begin{aligned} \mu_\Delta(\{(z, \ell) \in \hat{Y} : \varphi(z) > n\}) &= \sum_{q=1}^\infty \mu_\Delta(\{(z, \ell) \in \hat{Y} : \varphi(z) > n, \rho(z) = q\}) \\ &\leq \sum_{q=1}^\infty q \mu_Z(\{z \in Z : \varphi(z) > n, \rho(z) = q\}). \end{aligned}$$

For $k \geq 1$,

$$\begin{aligned}
\sum_{q=1}^{\infty} q\mu_Z(\varphi > n, \rho = q) &= \sum_{q=1}^k q\mu_Z(\varphi > n, \rho = q) + \sum_{q=k+1}^{\infty} q\mu_Z(\varphi > n, \rho = q) \\
&\leq k^2\mu_Z(\varphi > n) + \sum_{q=k+1}^{\infty} q\mu_Z(\rho = q) \\
&\ll k^2n^{-(\beta-\epsilon/2)} + \sum_{q=k+1}^{\infty} qe^{-cq} \ll k^2n^{-(\beta-\epsilon/2)} + e^{-ck/2},
\end{aligned}$$

where the implied constant is independent of k . Choosing $k = p \log n$ with p sufficiently large, we obtain the desired estimate. \blacksquare

Recall from Section 7 that $L_k : L^1(\Delta) \rightarrow L^1(\Delta)$ is the family of twisted transfer operators $L_k v = L(e^{ik \cdot \hat{h}} v)$ where $\hat{h} = h \circ \pi$ and L is the transfer operator corresponding to \hat{f} . From now on, with an obvious abuse of notation, we write $1_{\hat{Y}} L_k^n 1_{\hat{Y}}$ as a shorthand for $v \mapsto 1_{\hat{Y}} L_k^n (1_{\hat{Y}} v)$. We view these as operators $1_{\hat{Y}} L_k^n 1_{\hat{Y}} : F_{\theta}(\hat{Y}) \rightarrow L^1(\hat{Y})$.

Following Gou  zel [9, 10] (see also [2]), we define the sequences of operators

$$A_{k,n} : L^{\infty}(Z) \rightarrow L^1(\Delta), \quad B_{k,n} : F_{\theta}(\Delta) \rightarrow F_{\theta}(Z), \quad E_{k,n} : L^{\infty}(\Delta) \rightarrow L^1(\Delta),$$

as follows:

$$\begin{aligned}
(A_{k,n} v)(x) &= \sum_{\substack{\hat{f}^n z = x \\ z \in Z; \hat{f} z \notin Z, \dots, \hat{f}^n z \notin Z}} e^{g_n(z)} e^{ik \cdot \hat{h}_n(z)} v(z), \quad (B_{k,n} \hat{v})(z) = \sum_{\substack{\hat{f}^n u = z \\ u \notin Z, \dots, \hat{f}^{n-1} u \notin Z; \hat{f}^n u \in Z}} e^{g_n(u)} e^{ik \cdot \hat{h}_n(u)} \hat{v}(u), \\
(E_{k,n} \hat{v})(x) &= \sum_{\substack{\hat{f}^n u = x \\ u \notin Z, \dots, \hat{f}^n u \notin Z}} e^{g_n(u)} e^{ik \cdot \hat{h}_n(u)} \hat{v}(u).
\end{aligned}$$

As in [9, 10, 2],

$$L_k^n = \sum_{n_1+n_2+n_3=n} A_{k,n_1} T_{k,n_2} B_{k,n_3} + E_{k,n}, \quad (8.1)$$

and so

$$1_{\hat{Y}} L_k^n 1_{\hat{Y}} = \sum_{n_1+n_2+n_3=n} (1_{\hat{Y}} A_{k,n_1}) T_{k,n_2} (B_{k,n_3} 1_{\hat{Y}}) + 1_{\hat{Y}} E_{k,n} 1_{\hat{Y}}, \quad (8.2)$$

where

$$1_{\hat{Y}} A_{k,n} : L^{\infty}(Z) \rightarrow L^1(\hat{Y}), \quad B_{k,n} 1_{\hat{Y}} : F_{\theta}(\hat{Y}) \rightarrow F_{\theta}(Z), \quad 1_{\hat{Y}} E_{k,n} 1_{\hat{Y}} : L^{\infty}(\hat{Y}) \rightarrow L^1(\hat{Y}).$$

Proposition 8.4 *Uniformly in $k \in \mathbb{Z}^d$, $n \geq 1$,*

- (a) $\sum_{j \geq n} \|1_{\hat{Y}} A_{k,j}\|_{L^\infty(Z) \rightarrow L^1(\hat{Y})} = O(n^{-(\beta-\epsilon)}).$
- (b) $\|1_{\hat{Y}} E_{k,n} 1_{\hat{Y}}\|_{L^\infty(\hat{Y}) \rightarrow L^1(\hat{Y})} = O(n^{-(\beta-\epsilon)}).$
- (c) $\sum_{j \geq n} \|B_{k,j} 1_{\hat{Y}}\|_{F_\theta(\hat{Y}) \rightarrow F_\theta(Z)} = O(|k|^\epsilon n^{-(\beta-\epsilon)}).$

Proof (a) We have $|1_{\hat{Y}} A_{k,n} v|_\infty \leq |v|_\infty$ and $\text{supp } 1_{\hat{Y}} A_{k,n} v \subset \hat{Y} \cap \Delta_n$. Hence $|1_{\hat{Y}} A_{k,n} v|_1 \leq \mu_\Delta(\hat{Y} \cap \Delta_n) |v|_\infty$ and so $\|1_{\hat{Y}} A_{k,n}\|_{L^\infty(Z) \rightarrow L^1(\hat{Y})} \leq \mu_\Delta(\hat{Y} \cap \Delta_n)$. Part (a) now follows from Proposition 8.3.

Similarly $|1_{\hat{Y}} E_{k,n} 1_{\hat{Y}} \hat{v}|_\infty \leq |\hat{v}|_\infty$ and $\text{supp } 1_{\hat{Y}} E_{k,n} 1_{\hat{Y}} \hat{v} \subset \bigcup_{\ell > n} \hat{Y} \cap \Delta_\ell$. Hence $\|1_{\hat{Y}} E_{k,n} 1_{\hat{Y}}\|_{L^\infty(\hat{Y}) \rightarrow L^1(\hat{Y})} \leq \sum_{\ell > n} \mu_\Delta(\hat{Y} \cap \Delta_\ell)$, so part (b) follows from Proposition 8.3.

Finally,

$$(B_{k,n} 1_{\hat{Y}} \hat{v})(z) = \sum_{a \in \alpha} 1_{\{\varphi(a) > n\}} e^{g(z_a)} e^{ik \cdot \hat{h}_n(z_a, \varphi(a) - n)} 1_{\hat{Y}}(z_a, \varphi(a) - n) \hat{v}(z_a, \varphi(a) - n).$$

By Proposition 8.2, $1_{\hat{Y}}(z_a, \varphi(a) - n) = 1$ if and only if $a \times \{\varphi(a) - n\} \subset \hat{Y}$. Hence

$$(B_{k,n} 1_{\hat{Y}} \hat{v})(z) = \sum^* e^{g(z_a)} e^{ik \cdot h_n(f^{\varphi(a)-n} z_a)} \hat{v}(z_a, \varphi(a) - n),$$

where \sum^* denotes summation over those $a \in \alpha$ such that $a \times \{\varphi(a) - n\} \subset \hat{Y} \cap D_n$. By Proposition 8.2,

$$\sum^* \mu_Z(a) = \sum^* \mu_\Delta(a \times \{\varphi(a) - n\}) = \mu_\Delta(\hat{Y} \cap D_n). \quad (8.3)$$

Hence by (3.1), $|B_{k,n} 1_{\hat{Y}} \hat{v}|_\infty \leq C_3 |\hat{v}|_\infty \sum^* \mu_Z(a) \leq C_3 |\hat{v}|_\infty \mu_\Delta(\hat{Y} \cap D_n)$.

Also, for $z, z' \in Z$, we have that $(B_{k,n} 1_{\hat{Y}} \hat{v})(z) - (B_{k,n} 1_{\hat{Y}} \hat{v})(z') = I_1 + I_2 + I_3$, where

$$\begin{aligned} I_1 &= \sum^* (e^{g(z_a)} - e^{g(z'_a)}) e^{ik \cdot h_n(f^{\varphi(a)-n} z_a)} \hat{v}(z_a, \varphi(a) - n), \\ I_2 &= \sum^* e^{g(z'_a)} (e^{ik \cdot h_n(f^{\varphi(a)-n} z_a)} - e^{ik \cdot h_n(f^{\varphi(a)-n} z'_a)}) \hat{v}(z_a, \varphi(a) - n), \\ I_3 &= \sum^* e^{g(z'_a)} e^{ik \cdot h_n(f^{\varphi(a)-n} z'_a)} (\hat{v}(z_a, \varphi(a) - n) - \hat{v}(z'_a, \varphi(a) - n)). \end{aligned}$$

By (3.1) and (8.3), $|I_1| \leq C_3 |\hat{v}|_\infty \mu_\Delta(\hat{Y} \cap D_n) d_\theta(z, z')$, and

$$\begin{aligned} |I_3| &\leq C_3 |\hat{v}|_\theta \mu_\Delta(\hat{Y} \cap D_n) d_\theta(z_a, \varphi(a) - n, z'_a, \varphi(a) - n) \\ &= C_3 \theta |\hat{v}|_\theta \mu_\Delta(\hat{Y} \cap D_n) d_\theta(z, z'). \end{aligned}$$

Let $\gamma = \theta^{1/\epsilon}$. As in the proof of Proposition 3.2,

$$|h_n(f^{\varphi(a)-n} z_a) - h_n(f^{\varphi(a)-n} z'_a)| \leq \sum_{\ell=\varphi(a)-n}^{\varphi(a)-1} |h|_{C^\gamma} d(f^\ell z_a, f^\ell z'_a)^\gamma \ll n d_\gamma(z, z').$$

Hence using similar arguments as in the proof of Lemma 6.2,

$$\begin{aligned} |e^{ik \cdot h_n(f^{\varphi(a)-n} z_a)} - e^{ik \cdot h_n(f^{\varphi(a)-n} z'_a)}| &\leq 2|k|^\epsilon |h_n(f^{\varphi(a)-n} z_a) - h_n(f^{\varphi(a)-n} z'_a)|^\epsilon \\ &\ll |k|^\epsilon n^\epsilon d_\theta(z, z'). \end{aligned}$$

It follows that

$$|I_2| \ll |\hat{v}|_\infty |k|^\epsilon n^\epsilon \mu_\Delta(\hat{Y} \cap D_n) d_\theta(z, z').$$

Hence $|B_{k,n} 1_{\hat{Y}} \hat{v}|_\theta \ll |k|^\epsilon n^\epsilon \mu_\Delta(\hat{Y} \cap D_n) \|\hat{v}\|_\theta$ and so $\|B_{k,n} 1_{\hat{Y}}\|_{F_\theta(\hat{Y}) \rightarrow F_\theta(Z)} \ll |k|^\epsilon n^\epsilon \mu_\Delta(\hat{Y} \cap D_n)$. By Proposition 8.3, $\sum_{j \geq n} \|B_{k,j} 1_{\hat{Y}}\|_{F_\theta(\hat{Y}) \rightarrow F_\theta(Z)} = O(|k|^\epsilon n^{-(\beta-2\epsilon)})$, yielding part (c). \blacksquare

Corollary 8.5 *There exists $C, \xi > 0$ such that $\|1_{\hat{Y}} L_k^n 1_{\hat{Y}}\|_{F_\theta(\hat{Y}) \rightarrow L^1(\hat{Y})} \leq C |k|^\xi n^{-(\beta-\epsilon)}$ for all $k \in \mathbb{Z}^d \setminus \{0\}$, $n \geq 1$.*

Proof An elementary calculation shows that if u_n, v_n are real sequences and $|u_n| = O(n^{-\gamma})$, $\sum_{j \geq n} |v_j| = O(n^{-\gamma})$, where $\gamma > 0$, then $|(u \star v)_n| = O(n^{-\gamma})$. We apply this with $\gamma = \beta - \epsilon$.

Note that $G = \hat{f}^\varphi : Z \rightarrow Z$ is the first return map to Z for the tower map $\hat{f} : \Delta \rightarrow \Delta$. Also, the induced cocycle $H : Z \rightarrow \mathbb{R}$ is identical starting from f and h or from \hat{f} and \hat{h} so we still have nonexistence of approximate eigenfunctions when working in the tower set up. Hence Lemma 7.3 applies and we have that $\|T_{k,n}\| \ll |k|^\xi n^{-(\beta-\epsilon)}$.

Combining this with the estimates for $\sum_{j \geq n} 1_{\hat{Y}} A_{k,j}$ and $\sum_{j \geq n} B_{k,j} 1_{\hat{Y}}$ in Proposition 8.4, it follows that

$$\left\| \sum_{n_1+n_2+n_3=n} (1_{\hat{Y}} A_{k,n_1}) T_{k,n_2} (B_{k,n_3} 1_{\hat{Y}}) \right\|_{F_\theta(\hat{Y}) \rightarrow L^1(\hat{Y})} \ll |k|^{\xi+\epsilon} n^{-(\beta-\epsilon)}.$$

Using (8.2) and the estimate for $1_{\hat{Y}} E_{k,n} 1_{\hat{Y}}$ in Proposition 8.4, we obtain the desired estimate for $1_{\hat{Y}} L_k^n 1_{\hat{Y}}$. \blacksquare

Proof of Theorem 2.2 Since $\pi_* \mu_\Delta = \mu$ and v and w are supported in $Y \times \mathbb{T}^d$, for $k \in \mathbb{Z}^d \setminus \{0\}$ and $n \geq 1$,

$$\int_X e^{ik \cdot h_n} v_{-k} w_k \circ f^n d\mu = \int_\Delta e^{ik \cdot \hat{h}_n} \hat{v}_{-k} \hat{w}_k \circ \hat{f}^n d\mu_\Delta = \int_X 1_{\hat{Y}} L_k^n 1_{\hat{Y}} \hat{v}_{-k} \hat{w}_k d\mu_\Delta.$$

Hence

$$\left| \int_X e^{ik \cdot h_n} v_{-k} w_k \circ f^n d\mu \right| \leq |1_{\hat{Y}} L_k^n 1_{\hat{Y}} v_{-k}|_1 |w_k|_\infty \leq \|1_{\hat{Y}} L_k^n 1_{\hat{Y}}\| \|v_{-k}\|_\theta |w|_\infty.$$

By Corollary 8.5, $\|1_{\hat{Y}} L_k^n 1_{\hat{Y}}\| \ll |k|^\xi n^{-(\beta-\epsilon)}$. By Proposition 8.1, $\|v_{-k}\|_\theta \leq C \|v_{-k}\|_{C^\eta}$. It follows from the usual integration by parts argument that $\|v_{-k}\|_{C^\eta} \ll |k|^{-p} \|v\|_{C^{\eta,p}}$.

Hence

$$\left| \int_X e^{ik \cdot h_n} v_{-k} w_k \circ f^n d\mu \right| \ll |k|^{\xi-p} n^{-(\beta-\epsilon)} \|v\|_{C^{\eta,p}} |w|_\infty.$$

Taking $p > \xi + d$, we obtain that

$$|S_{v,w}(n)| \ll \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k|^{\xi-p} n^{-(\beta-\epsilon)} \|v\|_{C^{\eta,p}} |w|_{\infty} \ll n^{-(\beta-\epsilon)} \|v\|_{C^{\eta,p}} |w|_{\infty}$$

as required. ■

9 Proof of Theorem 2.3

Let $f : X \rightarrow X$ with induced map Gibbs-Markov map $G = f^{\varphi} : Z \rightarrow Z$ as in Section 3. Let μ_Z denote the associated ergodic G -invariant probability measure on μ . We suppose that $\mu_Z(\varphi > n) = O(n^{-\beta})$ where $\beta > 1$.

Again, we fix $\epsilon \in (0, \beta]$ such that $\beta - \epsilon$ is not an integer, $\varphi^{\epsilon} \in L^1(Z)$, and $\theta \in [\lambda^{-\eta^{\epsilon}}, 1)$. Also, we suppose that $\beta - \epsilon > 1$.

The tower map $\hat{f} : \Delta \rightarrow \Delta$, invariant measure μ_{Δ} , and lifted cocycle $\hat{h} = h \circ \pi : \Delta \rightarrow \mathbb{T}^d$ are all defined as before. Also we define $L : L^1(\Delta) \rightarrow L^1(\Delta)$ and $L_k \hat{v} = L(e^{ik \cdot \hat{h}} \hat{v})$ as before.

The arguments are similar to those in Section 8, the main differences being that we use (8.1) instead of (8.2) and that the estimates are simpler but weaker.

Proposition 9.1 *There is a constant $C > 0$ such that for all $k \in \mathbb{Z}^d \setminus \{0\}$, $n \geq 1$,*

$$\begin{aligned} \|A_{k,n}\|_{L^{\infty}(Z) \rightarrow L^1(\Delta)} &\leq \mu(\varphi > n), & \|B_{k,n}\|_{F_{\theta}(\Delta) \rightarrow F_{\theta}(Z)} &\leq C \mu(\varphi > n) |k|^{\epsilon} n^{\epsilon}, \\ \|E_{k,n}\|_{L^{\infty}(\Delta) \rightarrow L^1(\Delta)} &\leq \sum_{j > n} \mu(\varphi > j). \end{aligned}$$

Proof These estimates are similar to the ones in Proposition 8.4. ■

Corollary 9.2 *Assume condition (v). There exists $C, \xi > 0$ such that $\|L_k^n\|_{F_{\theta}(\Delta) \rightarrow L^1(\Delta)} \leq C |k|^{\xi} n^{-(\beta-1)}$ for all $k \in \mathbb{Z}^d \setminus \{0\}$, $n \geq 1$,*

Proof We estimate the sequences in (8.1). As in the proof of Corollary 8.5, $\|T_{k,n}\| \ll |k|^{\xi} n^{-(\beta-\epsilon)}$. By Proposition 9.1, the same estimate holds for $\|A_{k,n}\|$ and $\|B_{k,n}\|$. Since $\beta - \epsilon > 1$, the convolution of these three sequences is also $O(|k|^{\xi} n^{-(\beta-\epsilon)})$ for some ξ . Finally, by Proposition 9.1, $\|E_{k,n}\| \ll n^{-(\beta-1)}$. ■

Proof of Theorem 2.3 This follows from Corollary 9.2 in the same way that Theorem 2.2 followed from Corollary 8.5. ■

A Proof of Proposition 4.6

We recall the “good asymptotics” construction in [6, 7]. Let $p_0 \in Z_0$ be a fixed point for G . As in [6, 7], we construct a sequence of N -periodic points p_N , $N \geq 1$, for G with orbits lying in Z_0 . The entire construction is done in a neighborhood of one homoclinic trajectory for p_0 , so any cocycle $H : Z \rightarrow \mathbb{T}^d$ can be lifted to a cocycle with values in \mathbb{R}^d . The upshot is that there exists $\gamma \in (0, 1)$ (depending only on $(dG)_p$) such that for any C^1 cocycle $H : Z \rightarrow \mathbb{R}^d$, there exist κ , $J_N \in \mathbb{R}^d$, such that

$$H_N(p_N) = NH(p_0) + \kappa + J_N \gamma^N + o(\gamma^N) \quad \text{as } N \rightarrow \infty, \quad (\text{A.1})$$

Here the i 'th coordinate of J_N has the form $J_{N,i} = E_{N,i} \cos(N\theta_i + \psi_{N,i})$, $i = 1, \dots, d$, where $E_{N,i}$ is a bounded sequence of real numbers and either (a) $\theta_i = 0$ and $\psi_{N,i} \equiv 0$ or (b) $\theta_i \in (0, \pi)$ and $\psi_{N,i} \in (\tilde{\theta}_i - \pi/12, \tilde{\theta}_i + \pi/12)$ for some $\tilde{\theta}_i$.

The cocycle H has *good asymptotics* [7] if $\liminf_{N \rightarrow \infty} |E_{N,i}| > 0$ for each i . By [6, 7], for any $r \geq 2$ there is a C^2 open and C^r dense set of C^r cocycles h such that H has good asymptotics.

Proof of Proposition 4.6 By the above, it suffices to show that if H has good asymptotics, then there are no approximate eigenfunctions on Z_∞ .

Suppose for contradiction that there are approximate eigenfunctions u_j on Z_∞ , so $|M_{k_j, \omega_j}^{n_j} u_j - e^{i\chi_j} u_j| = O(|k_j|^{-\xi})$. To complete the proof we show that $\liminf_{n \rightarrow \infty} |E_{N,i}| = 0$ for some $i \in \{1, \dots, d\}$.

Note that since φ is integer-valued, there exists $\kappa' \in \mathbb{Z}$ such that

$$\varphi_N(p_N) = N\varphi(p_0) + \kappa', \quad N \geq 1. \quad (\text{A.2})$$

The remainder of the argument is an adaptation of [7, Proof of Theorem 1.6(a)]. Since $|M_{k_j, \omega_j}|_\infty = 1$, it is immediate that for all $N \geq 1$,

$$|e^{-i k_j H_{n_j N}} e^{-i \omega_j \varphi_{n_j N}} u_j \circ G^{n_j N} - e^{i N \chi_j} u_j| = |M_{k_j, \omega_j}^{n_j N} u_j - e^{i N \chi_j} u_j| = O(N |k_j|^{-\xi}).$$

Substituting in the periodic points p_N , and using the fact that $|u_j| \equiv 1$, we obtain

$$|e^{i(n_j k_j \cdot H_N(p_N) + n_j \omega_j \varphi_N(p_N) + N \chi_j)} - 1| = O(N |k_j|^{-\xi}),$$

and hence

$$\text{dist}(n_j k_j \cdot H_N(p_N) + n_j \omega_j \varphi_N(p_N) + N \chi_j, 2\pi\mathbb{Z}) = O(N |k_j|^{-\xi}).$$

Similarly,

$$\text{dist}(N n_j k_j \cdot H(p_0) + N n_j \omega_j \varphi(p_0) + N \chi_j, 2\pi\mathbb{Z}) = O(N |k_j|^{-\xi}),$$

Subtracting these expressions and using (A.1) and (A.2),

$$\text{dist}(n_j k_j \cdot (\kappa + J_N \gamma^N + o(\gamma^N)) + n_j \omega_j \kappa', 2\pi\mathbb{Z}) = O(N |k_j|^{-\xi}).$$

Recall that $n_j = \lceil \zeta \ln |k_j| \rceil$. Set $N = N(j) = \lceil \rho \ln |k_j| \rceil$. For large enough $\rho > 0$, we have $n_j k_j E_{N(j)} \gamma^{N(j)} = O(|k_j|^{-2\xi})$. It follows that $\text{dist}(n_j k_j \cdot \kappa + n_j \omega_j \kappa', 2\pi\mathbb{Z}) = O(|k_j|^{-\xi} \ln |k_j|)$ and so

$$\text{dist}(n_j k_j \cdot (J_N \gamma^N + o(\gamma^N)), 2\pi\mathbb{Z}) = O(N|k_j|^{-\xi}) + O(|k_j|^{-\xi} \ln |k_j|). \quad (\text{A.3})$$

Let $S = \sup_N |J_N|$ and set $M(j) = \lceil (\ln(n_j |k_j|) + \ln S + \ln 2) / (-\ln \gamma) \rceil + 1$. Then $S n_j |k_j| \gamma^{M(j)} = \frac{1}{2} \gamma^{\rho_j}$, with $\rho_j \in (0, 1]$. In particular, $|S n_j |k_j| \gamma^{M(j)}| \leq \frac{1}{2}$ and so taking $N = M(j) + m$ with $m \in \mathbb{N}$ fixed, condition (A.3) implies that

$$\lim_{j \rightarrow \infty} n_j k_j \cdot J_{M(j)+m} \gamma^{M(j)} = 0.$$

Moreover, $n_j |k_j| \gamma^{M(j)} \geq \gamma / (2S)$ and it follows that there exists $i \in \{1, \dots, d\}$ such that

$$\lim_{j \rightarrow \infty} E_{M(j)+m,i} \cos((M(j) + m)\theta_i + \psi_{M(j)+m,i}) = 0.$$

We show that for this i , there is a choice of $m \in \mathbb{N}$ for which $\cos((M(j) + m)\theta_i + \psi_{M(j)+m,i})$ does not converge to 0 as $j \rightarrow \infty$

Assume for contradiction that for each integer $m \geq 0$

$$\lim_{j \rightarrow \infty} (M(j) + m)\theta_i + \psi_{M(j)+m,i} = \pi/2 \pmod{\pi}. \quad (\text{A.4})$$

Recall that if $\theta_i = 0$ then $\psi_N \equiv 0$, hence (A.4) fails (with $m = 0$). Otherwise, $\theta_i \in (0, \pi)$ and $|\psi_N - \tilde{\theta}_i| < \pi/12$. Taking differences of (A.4) for various values of m we obtain that $\ell\theta_i \in [-\pi/6, \pi/6] \pmod{\pi}$ for all ℓ , which is impossible. \blacksquare

B Proof of Proposition 7.2

In this appendix, we show that the coefficients $T_{k,n}$ and $\hat{T}_{k,n}$ of T_k coincide for all $\beta > 0$, $k \in \mathbb{Z}^d \setminus \{0\}$, $n \geq 0$. The case $k = 0$ was treated in [18] using a dominated convergence argument on an annulus at the boundary of the unit disk. Here we use the same strategy, but the details are somewhat different.

Throughout we assume nonexistence of eigenfunctions, and we work with a fixed $k \in \mathbb{Z}^d \setminus \{0\}$. Also, we fix $\epsilon \in (0, 1]$ such that $\varphi^\epsilon \in L^1(Z)$.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. First, we extend the definition of R_k to the closed unit disk, setting $R_k(z) = \sum_{n=1}^{\infty} R_{k,n} z^n$ for all $z \in \overline{\mathbb{D}}$. Then $R_k(z)v = R(e^{ik \cdot H} z^\varphi v)$. Note that $R_k(e^{i\omega})$ coincides with the operator previously denoted $R_k(\omega)$.

Proposition B.1 $\sup_{\omega \in [0, 2\pi]} \|(I - R_k(e^{i\omega}))^{-1}\|_\theta < \infty$.

Proof A standard consequence (see for example [12]) of Proposition 6.1(b) and Corollary 6.3(b) is that $R_k(e^{i\omega})$ has essential spectral radius at most θ . Hence if $1 \in \text{spec } R_k(e^{i\omega})$, then there exists a nonzero function $v \in F_\theta(Z)$ such that $R_k(e^{i\omega})v = v$. A calculation using the fact that $M_{k,\omega}$ is the L^2 adjoint of $R_k(e^{i\omega})$ (see for example [17, p. 429]) shows that $M_{k,\omega}v = v$ contradicting the assumption that there are no eigenfunctions.

Hence $1 \notin \text{spec } R_k(e^{i\omega})$, and so $\|(I - R_k(e^{i\omega}))^{-1}\|_\theta < \infty$, for each $\omega \in [0, 2\pi]$. By Corollary 6.4, $\omega \mapsto R_k(e^{i\omega})$ is continuous and the result follows. \blacksquare

Remark B.2 Under the assumption that there are no approximate eigenfunctions, we could bypass Proposition B.1 and simply quote Lemma 6.8.

The next step is to extend this estimate to an annulus.

Proposition B.3 *There exists $C \geq 1$ such that $\|R_k(e^{i\omega}) - R_k(\rho e^{ik\omega})\|_\theta \leq C(1 - \rho)^\epsilon$, for all $\rho \in [0, 1]$, $\omega \in [0, 2\pi]$.*

Proof Define $S_{\omega,\rho} = R_k(e^{i\omega}) - R_k(\rho e^{ik\omega})$. Let $v \in F_\theta(Z)$. Then

$$S_{\omega,\rho}v = R(e^{ik \cdot H} e^{i\omega\varphi}(1 - \rho^\varphi))v.$$

Hence in the usual notation, for $z \in Z$,

$$(S_{\omega,\rho}v)(z) = \sum_{a \in \alpha} e^{g(z_a)} e^{ik \cdot H(z_a)} e^{i\omega\varphi(a)} (1 - \rho^{\varphi(a)}) v(z_a).$$

By (3.1),

$$|S_{\omega,\rho}v|_\infty \leq C_3 |v|_\infty \sum_{a \in \alpha} \mu_Z(a) (1 - \rho^{\varphi(a)}).$$

Now $1 - \rho^n \leq \min\{1, (1 - \rho)n\} \leq (1 - \rho)^\epsilon n^\epsilon$. Hence,

$$|S_{\omega,\rho}v|_\infty \leq C_3 |v|_\infty \sum_{a \in \alpha} \mu_Z(a) (1 - \rho)^\epsilon \varphi(a)^\epsilon = C_3 |\varphi|_1 |v|_\infty (1 - \rho)^\epsilon \ll |v|_\infty (1 - \rho)^\epsilon.$$

Next, for $z, z' \in Z$,

$$|(S_{\omega,\rho}v)(z) - (S_{\omega,\rho}v)(z')| \leq I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \sum_{a \in \alpha} (e^{g(z_a)} - e^{g(z'_a)}) e^{ik \cdot H(z_a)} e^{i\omega\varphi(a)} (1 - \rho^{\varphi(a)}) v(z_a), \\ I_2 &= \sum_{a \in \alpha} e^{g(z'_a)} (e^{ik \cdot H(z_a)} - e^{ik \cdot H(z'_a)}) e^{i\omega\varphi(a)} (1 - \rho^{\varphi(a)}) v(z_a), \\ I_3 &= \sum_{a \in \alpha} e^{g(z'_a)} e^{ik \cdot H(z'_a)} e^{i\omega\varphi(a)} (1 - \rho^{\varphi(a)}) (v(z_a) - v(z'_a)). \end{aligned}$$

Using estimates as in the proof of Lemma 6.2 combined with the argument above for estimating $1 - \rho^{\varphi(a)}$, we obtain

$$\begin{aligned} |I_1| &\leq C_3 |\varphi^\epsilon|_1 |v|_\infty (1 - \rho)^\epsilon d_\theta(z, z'), \quad |I_3| \leq C_3 |\varphi^\epsilon|_1 |v|_\theta (1 - \rho)^\epsilon d_\theta(z, z'), \\ |I_2| &\leq 2C_2 C_3 |k|^\epsilon |h|_{C^\eta}^\epsilon |\varphi^\epsilon|_1 |v|_\infty (1 - \rho)^\epsilon d_\theta(z, z'). \end{aligned}$$

Hence $|S_{\omega, \rho} v|_\theta \ll \|v\|_\theta (1 - \rho)^\epsilon$ and the result follows. \blacksquare

Corollary B.4 *There exists $\rho_0 \in (0, 1]$ such that $\sup_{\rho \in [\rho_0, 1]} \sup_{\omega \in [0, 2\pi]} \|(I - R_k(\rho e^{i\omega}))^{-1}\|_\theta < \infty$.*

Proof We use the resolvent identity

$$(I - R_k(\rho e^{i\omega}))^{-1} = (I - R_k(e^{i\omega}))^{-1} (I + A_{\omega, \rho})^{-1}, \quad (\text{B.1})$$

where

$$A_{\omega, \rho} = (R_k(e^{i\omega}) - R_k(\rho e^{i\omega}))(I - R_k(e^{i\omega}))^{-1},$$

By Propositions B.1 and B.3, $\|A_{\omega, \rho}\|_\theta \ll (1 - \rho)^\epsilon$ for all $\rho \in [0, 1]$, $\omega \in [0, 2\pi]$. Hence we can choose ρ_0 so that $\|A_{\omega, \rho}\|_\theta \leq \frac{1}{2}$ for all $\rho \in [\rho_0, 1]$, $\omega \in [0, 2\pi]$. It follows that $\|(I + A_{\omega, \rho})^{-1}\|_\theta \leq 2$. The result follows from (B.1) and Proposition B.1. \blacksquare

Next, we define $T_k(z) = \sum_{n=0}^{\infty} T_{k,n} z^n$. Since $|T_{k,n}|_1 \leq 1$ for all n , the family $T_k(z)$ is analytic on the open unit disk \mathbb{D} when viewed as a family of operators on $L^1(Z)$. Hence it is certainly analytic as a family of operators from $F_\theta(Z)$ to $L^1(Z)$.

The renewal equation becomes $T_k(z) = (I - R_k(z))^{-1}$ for $z \in \mathbb{D}$. By Corollary B.4, we can extend $T_k(z)$ to $\overline{\mathbb{D}}$ as a continuous family of operators from $F_\theta(Z)$ to $L^1(Z)$.

The Fourier coefficients of $T_k : S^1 \rightarrow L(F_\theta(Z), L^1(Z))$ are given by $\hat{T}_{k,n} = (2\pi)^{-1} \int_0^{2\pi} T_k(e^{i\omega}) e^{-in\omega} d\omega$. Also the coefficients of the analytic function $T_k : \mathbb{D} \rightarrow L(F_\theta(Z), L^1(Z))$ are given by $T_{k,n} = (2\pi)^{-1} \int_0^{2\pi} \rho^{-n} T_k(\rho e^{i\omega}) e^{-in\omega} d\omega$ for any $\rho \in (0, 1]$. By Corollary B.4 and the renewal equation, the integrand $I_\rho(\omega) = \rho^{-n} T_k(\rho e^{i\omega}) e^{-in\omega}$ satisfies the uniform bound $\sup_{\rho \in [\rho_0, 1]} \sup_{\omega \in [0, 2\pi]} \|I_\rho(\omega)\|_{F_\theta(Z) \rightarrow L^1(Z)} < \infty$. Letting $\rho \rightarrow 1^-$, it follows from the dominated convergence theorem that $T_{k,n} = \hat{T}_{k,n}$ as required.

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References

- [1] J. Aaronson. *An Introduction to Infinite Ergodic Theory*. Math. Surveys and Monographs **50**, Amer. Math. Soc., 1997.

- [2] H. Bruin, M. Holland and I. Melbourne. Subexponential decay of correlations for compact group extensions of nonuniformly expanding systems. *Ergodic Theory Dynam. Systems* **25** (2005) 1719–1738.
- [3] H. Bruin and D. Terhesiu. Upper and lower bounds for the correlation function via inducing with general return times. Preprint, 2015.
- [4] D. Dolgopyat. Prevalence of rapid mixing in hyperbolic flows. *Ergodic Theory Dynam. Systems* **18** (1998) 1097–1114.
- [5] D. Dolgopyat. On mixing properties of compact group extensions of hyperbolic systems. *Israel J. Math.* **130** (2002) 157–205.
- [6] M. J. Field, I. Melbourne and A. Török. Stable ergodicity for smooth compact Lie group extensions of hyperbolic basic sets. *Ergodic Theory Dynam. Systems* **25** (2005) 517–551.
- [7] M. J. Field, I. Melbourne and A. Török. Stability of mixing and rapid mixing for hyperbolic flows. *Ann. of Math.* **166** (2007) 269–291.
- [8] S. Gouëzel. Sharp polynomial estimates for the decay of correlations. *Israel J. Math.* **139** (2004) 29–65.
- [9] S. Gouëzel. Vitesse de décorrélation et théorèmes limites pour les applications non uniformément dilatantes. Ph. D. Thesis. Ecole Normale Supérieure, 2004.
- [10] S. Gouëzel. Berry-Esseen theorem and local limit theorem for non uniformly expanding maps. *Ann. Inst. H. Poincaré Probab. Statist.* **41** (2005) 997–1024.
- [11] S. Gouëzel. Correlation asymptotics from large deviations in dynamical systems with infinite measure. *Colloq. Math.* **125** (2011) 193–212.
- [12] H. Hennion. Sur un théorème spectral et son application aux noyaux lipchitziens. *Proc. Amer. Math. Soc.* **118** (1993) 627–634.
- [13] H. Hu. Decay of correlations for piecewise smooth maps with indifferent fixed points. *Ergodic Theory Dynam. Systems* **24** (2004) 495–524.
- [14] Y. Katznelson. *An Introduction to Harmonic Analysis*. Dover, New York, 1976.
- [15] C. Liverani, B. Saussol and S. Vaienti. A probabilistic approach to intermittency. *Ergodic Theory Dynam. Systems* **19** (1999) 671–685.
- [16] I. Melbourne. Rapid decay of correlations for nonuniformly hyperbolic flows. *Trans. Amer. Math. Soc.* **359** (2007) 2421–2441.
- [17] I. Melbourne and M. Nicol. Statistical properties of endomorphisms and compact group extensions. *J. London Math. Soc.* **70** (2004) 427–446.

- [18] I. Melbourne and D. Terhesiu. Operator renewal theory and mixing rates for dynamical systems with infinite measure. *Invent. Math.* **189** (2012) 61–110.
- [19] I. Melbourne and D. Terhesiu. Decay of correlations for nonuniformly expanding systems with general return times. *Ergodic Theory Dynam. Systems* **34** (2014) 893–918.
- [20] Y. Pomeau and P. Manneville. Intermittent transition to turbulence in dissipative dynamical systems. *Comm. Math. Phys.* **74** (1980) 189–197.
- [21] O. M. Sarig. Subexponential decay of correlations. *Invent. Math.* **150** (2002) 629–653.
- [22] D. Terhesiu. Improved mixing rates for infinite measure preserving systems. *Ergodic Theory Dynam. Systems* (2015) 585–614.
- [23] M. Thaler. Estimates of the invariant densities of endomorphisms with indifferent fixed points. *Israel J. Math.* **37** (1980) 303–314.
- [24] L.-S. Young. Recurrence times and rates of mixing. *Israel J. Math.* **110** (1999) 153–188.
- [25] R. Zweimüller. Ergodic structure and invariant densities of non-Markovian interval maps with indifferent fixed points. *Nonlinearity* **11** (1998) 1263–1276.
- [26] R. Zweimüller. Ergodic properties of infinite measure-preserving interval maps with indifferent fixed points. *Ergodic Theory Dynam. Systems* **20** (2000) 1519–1549.